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A Note on Factors for Absolute Norlund Summability

W.T. Sulaiman

Abstract. Improvement and generalization for two known results concerning summability factors for absolute Norlund summability of infinite series is presented.

1. **Introduction**

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and let $r_n = na_n$. By u_n^α and t_n^α we denote n-th Cesaro means of order $\alpha > -1$ of the sequences (s_n) and (r_n) respectively. These are

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}s_\nu\,, \tag{1.1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\nu a_\nu\,, \tag{1.2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1,\ A_0^\alpha = 1,\ A_{-n}^\alpha = 0.$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \ge 1$, if (see [5], [7])

$$\sum_{n=1}^{\infty} n^{k-1}|\Delta u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty}\frac{1}{n}|t_n^\alpha|^k < \infty\,, \tag{1.3}$$

where $\Delta u_n = u_n - u_{n+1}$. $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability on taking $\alpha = 1$. The series $\sum a_n$ is said to be summable $\varphi - |C,\alpha|_k$, $k \ge 1$, if (see [10])

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}|t_n^\alpha|^k < \infty\,. \tag{1.4}$$

$\varphi - |C,\alpha|_k$ summability reduces to $|C,\alpha|_k$ summability by taking $\varphi = n$.

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Let (p_n) be a sequence of constants, real or complex, and we write

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \text{ as } n \to \infty, \ n \geq 0.$$

The series $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if (see[8])

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta \sigma_{n-1}|^k < \infty \tag{1.5}$$

where

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v. \tag{1.6}$$

In the special case when

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \quad \alpha \geq 0,$$

$|N, p_n|_k$ summability reduces to $|C, \alpha|_k$ summability. The series $\sum a_n$ is said to be summable $\varphi - |N, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |\Delta \sigma_{n-1}|^k < \infty. \tag{1.7}$$

In the special case when $\varphi = n$, $\varphi - |N, p_n|_k$ summability reduces to $|N, p_n|_k$ summability.

2. Known Results

Theorem 2.1 ([6]). *Let (p_n) be a non-increasing sequences. If $\sum a_n$ is summable $|C, 1|_k$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.*

Theorem 2.2 ([11]). *Let (φ_n) be a sequence of positive real numbers with (λ_n) satisfying the following*

$$\sum_{v=1}^{m} \frac{\varphi_v^{k-1}}{v^k} |t_v|^k = O(\log m) \ \text{as} \ m \to \infty, \tag{2.1}$$

$$\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{k+1}} = O\left(\frac{\varphi_v^{k-1}}{v^k}\right), \tag{2.2}$$

$$\lambda_m = o(1) \quad \text{as} \quad m \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{m} n \log n |\Delta^2 \lambda_n| = O(1), \tag{2.4}$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - |C, 1|_k$, $k \geq 1$.

Theorem 2.3 ([2]). *Let (p_n) be a non-increasing sequence such that $p_0 > 0$, $p_n \geq 0$, and let (X_n) be a positive non-decreasing sequence satisfying*

$$|\lambda_n| X_n = O(1) \quad \text{as} \quad n \to \infty \tag{2.5}$$

$$\sum_{n=1}^{\infty} n |\Delta^2 \lambda_n| X_n < \infty. \tag{2.6}$$

If the sequence (w_n^α) *defined by*

$$w_n^\alpha = \begin{cases} |t_n^\alpha|, \alpha = 1 \\ \max_{1\le v\le n} |t_n^\alpha|, \quad 0<\alpha<1 \end{cases} \tag{2.7}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-1}(w_n^\alpha)^k = O(X_m) \quad as \quad m\to\infty, \tag{2.8}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N,p_n|_k$, $k\ge 1$, $0<\alpha\le 1$.

3. **Lemmas**

The following Lemmas are needed for our aim

Lemma 3.1 ([4])**.** *If* $0<\alpha\le 1$ *and* $1\le v\le n$, *then*

$$\left|\sum_{\rho=0}^{v} A_{n-\rho}^{\alpha-1} a_\rho\right| \le \max_{1\le m\le v}\left|\sum_{\rho=0}^{v} A_{m-\rho}^{\alpha-1} a_\rho\right| \tag{3.1}$$

Lemma 3.2 ([1])**.** *Under the conditions on* (X_n) *and* (λ_n) *as taken in the statement of Theorem* 3*, the following conditions holds*

$$nX_n|\Delta\lambda_n| = O(1) \ \ as \ \ n\to\infty \tag{3.2}$$

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| X_n < \infty. \tag{3.3}$$

Lemma 3.3 ([9])**.** *If* $-1<\alpha\le\beta$, $k>1$ *and the series* $\sum a_n$ *is summable* $|C,\alpha|_k$, *then it is summable* $|C,\beta|_k$.

Lemma 3.4. *The condition* (4.1) *is weaker than*

$$\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k}(w_n^\alpha)^k = O(X_m). \tag{3.4}$$

Proof. If (3.4) holds, then we have

$$\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k X_n^{k-1}}(w_n^\alpha)^k = O\left(\frac{1}{X_1^{k-1}}\right)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}(w_n^\alpha)^k = O(X_m).$$

while if (4.1) is satisfied then,

$$\begin{aligned}\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k}(w_n^\alpha)^k &= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k X_n^{k-1}}(w_n^\alpha)^k X_n^{k-1} \\ &= \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{\varphi_v^{k-1}}{v^k X_v^{k-1}}(w_v^\alpha)^k\right)\Delta X_n^{k-1} + \left(\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k X_n^{k-1}}(w_n^\alpha)^k\right)X_m^{k-1}\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=1}^{m-1} X_n|\Delta X_n^{k-1}| + O(X_m)X_m^{k-1} \\
&= O(X_{m-1})\sum_{n=1}^{m-1}(X_{n+1}^{k-1} - X_n^{k-1}) + O(X_m^k) \\
&= O(X_{m-1})(X_m^{k-1} - X_1^{k-1}) + O(X_m^k) \\
&= O(X_m^k).
\end{aligned}
$$

□

Therefore (3.4) implies (4.1) but not conversely.

The object of this paper is to present a general result not only covering Theorems 2 and 3, but as well to obtain an improvements for them. In fact we give the following theorem:

4. Main Result

Theorem 4.1. *Let* (X_n) *be a positive non-decreasing sequence. If the conditions* (1.12) *and* (1.13) *are satisfied and if the sequence* (w_n^α) *defined by* (1.14) *satisfies*

$$\sum_{n=1}^{m} \frac{\varphi_n^{k-1}(w_n^\alpha)^k}{n^k X_n^{k-1}} = O(X_m) \tag{4.1}$$

and

$$\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{k+\alpha k}} = O\left(\frac{\varphi_v^{k-1}}{v^{k+\alpha k-1}}\right), \tag{4.2}$$

then the series $\sum a_n\lambda_n$ *is summable* $\varphi - |C,\alpha|_k$, $k \geq 1$, $0 < \alpha \leq 1$.

Remark 4.1. For the special case $\alpha = 1$, Theorem 8 gives an improvement of Theorem 2 in the sense that conditions (4.1) and (4.2) for $\alpha = 1$, $X_n = \log n$ are both weaker than conditions (1.8) and (1.9), respectively.

Remark 4.2. For the special case $\varphi = n$, Theorem 8 gives an improvement of Theorem 3 in the sense that condition (4.1) for $\varphi = n$, is weaker than condition (1.15). That is Theorem 3 follows from Theorem 8 by putting $\varphi = n$, and then making use of Lemma 6 and Theorem 1.

5. Proof of Theorem 8

Let (T_n^α) be the n-th (C,α), $(0 < \alpha \leq 1)$ mean of the sequence $(na_n\lambda_n)$. Then, we have

$$
\begin{aligned}
T_n^\alpha &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v \\
&= \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{r=1}^{v} A_{n-r}^{\alpha-1} r a_r + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v\,.
\end{aligned}
$$

By Lemma 3.1, the above implies

$$\begin{aligned}|T_n^\alpha| &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta\lambda_v| + |\lambda_n| w_n^\alpha \\ &= T_{n1} + T_{n2}\,.\end{aligned}$$

In order to complete the proof, it is sufficient, by Minkowski's inequality to show that

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |T_{nj}|^k < \infty\,, \quad j = 1, 2.$$

Now applying Holder's inequality,

$$\begin{aligned}\sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^k}|T_{n1}|^k &= \sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^k}\left(\frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta\lambda_v|\right)^k \\ &\le \sum_{n=1}^{m+1} \frac{\varphi_n^{k-1}}{n^k}\frac{1}{(A_n^\alpha)^k}\sum_{v=1}^{n-1}(A_v^\alpha)^k (w_v^\alpha)^k |\Delta\lambda_v| X_v^{1-k}\left(\sum_{v=1}^{n-1}|\Delta\lambda_v| X_v\right)^{k-1} \\ &= O(1)\sum_{n=2}^{m+1}\frac{\varphi_n^{k-1}}{n^{k+\alpha k}}\sum_{v=1}^{n-1} v^{\alpha k}(w_v^\alpha)^k |\Delta\lambda_v| X_v^{1-k} \\ &= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^\alpha)^k |\Delta\lambda_v| X_v^{1-k}\sum_{n=v+1}^{m+1}\frac{\varphi_n^{k-1}}{n^{k+\alpha k}} \\ &= O(1)\sum_{v=1}^{m} v|\Delta\lambda_v|\frac{\varphi_v^{k-1}(w_v^\alpha)^k}{v^k X_v^{k-1}} \\ &= O(1)\sum_{v=1}^{m-1}|\Delta(v\Delta\lambda_v)|\sum_{r=1}^{v}\frac{\varphi_r^{k-1}(w_r^\alpha)^k}{r^k X_r^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{\varphi_v^{k-1}(w_v^\alpha)^k}{v^k X_v^{k-1}} \\ &= O(1)\sum_{v=1}^{m}|\Delta\lambda_v| X_v + O(1)\sum_{v=1}^{m}(v+1)|\Delta^2\lambda_v| X_v + O(1)m|\Delta\lambda_m| X_m \\ &= O(1),\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}|T_{n2}|^k &= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}(|\lambda_n| w_n^\alpha)^k \\ &= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k X_n^{k-1}}(|\lambda_n| X_n)^{k-1}(w_n^\alpha)^k|\lambda_n| \\ &= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}(w_n^\alpha)^k}{n^k X_n^{k-1}}|\lambda_n| \\ &= O(1)\sum_{n=1}^{m-1}|\Delta|\lambda_n||\sum_{v=1}^{n}\frac{\varphi_v^{k-1}(w_v^\alpha)^k}{v^k X_v^{k-1}} + O(1)|\lambda_m|\sum_{n=1}^{m}\frac{\varphi_n^{k-1}(w_n^\alpha)^k}{n^k X_n^{k-1}}\end{aligned}$$

$$= O(1)\sum_{n=1}^{m} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m$$
$$= O(1).$$

The proof is complete.

Theorem 5.1. *If the conditions of Theorem* 8 *are satisfied and if* $\psi_x = \psi(x)$ *is a convex function, with* $\psi(0) = 0$*, then the series* $\sum n a_n \lambda_n / \psi_n$ *is summable* $\varphi - |C, \alpha|_k$*,* $k \geq 1$*,* $0 < \alpha \leq 1$.

Proof. The proof follows exactly as it has been done in Theorem 8 noticing that $n/\psi_n = O(1)$, as $\psi(x)/x$ is non-decreasing. □

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W.T. Sulaiman, *Department of Computer Engineering, College of Engineering, University of Mosul, Iraq*.
E-mail: `waadsulaiman@hotmail.com`

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