Communications in Mathematics and Applications
Vol. 9, No. 4, pp. 737–745, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v9i4.1197

Research Article

A New Type of Ideal Convergence of Difference Sequence in Probabilistic Normed Space

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Abstract. The idea of difference sequence sets $X(\Delta)=\{x=(x_k):\Delta x\in X\}$ with $X=l_\infty$, c and c_0 was introduced by Kizmaz [10]. Mursaleen and Mohiuddine [13] defined the idea of *probabilistic normed space* (PNS) and the ideal convergence in PNS. Motivated by the above two concepts, we in this paper introduce the notion of difference I-convergent sequence in PNS and study the elementary properties of this convergence.

Keywords. Triangular norm; Probabilistic normed space; ΔI-convergence; ΔI^*-convergence; ΔI-limit points; ΔI-cluster points

MSC. Primary 40A05; Secondary 46A70

Received: March 1, 2018 **Accepted:** April 5, 2018

This article was submitted to "*5th International Conference on Recent Advances in Pure and Applied Mathematics*" (ICRAPAM 2018) organized by Karadeniz Technical University Prof. Dr. Osman Turan Convention Center in Trabzon, Turkey during July 23-27, 2018.

Academic Editor: Prof. Ekrem Savas, Istanbul Ticaret University, Turkey

1. Introduction

Statistical convergence for real sequences was defined by Fast [4] and Steinhaus [17] in 1951 and this concept was studied and applied by many authors, namely [6], [3]. This idea was generalised to I-convergence by Kostyrko *et al*. [11]. Lately, I-convergence for sequence of functions has been studied by Balcerzak *et al*. [2].

The idea of probabilistic normed spaces is the generalisation of normed space and has emerged from the concept of statistical metric spaces which was defined by Menger [12] and later on was studied by Schweizer and Sklar [16]. It is useful in many areas like continuity properties[1], topological spaces [5] etc. In 2012, M. Mursaleen and S.A. Mohiuddine gave the concept of ideal convergence in *probabilistic normed space* (PNS) [13]. Difference sequence spaces $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ were defined by H. Kizmaz [10].

In this paper, we have defined the notion of difference ideal convergence of sequences in probabilistic normed space. Section 2 deals with the idea of ΔI-and ΔI^*-PNS and some basic algebraic properties of these notions. In section 3, we have defined ΔI-limit points and ΔI-cluster points in PNS and studied some related results.

To recall certain definitions such as ideal, I-convergence, solid space, sequence algebra, difference convergence in probabilistic normed space. etc which will be used throughout this paper, we refer to [11], [7], [9], [15], [8], [14], [14], [18]. We denote by $\mathbb{N}$, $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}_0^+$ as the set of natural numbers, complex numbers, real numbers and positive real numbers.

2. Main Results

In this section, we study the concept ΔI-and ΔI^*-convergence of sequence in probabilistic normed space. We take ideal I as non-trivial admissible ideal.

2.1 ΔI-Convergence in PNS

Definition 2.1. Consider an ideal I in $\mathbb{N}$ and $(X,\nu,*)$ be a probabilistic normed space. A sequence $x=(x_k)\in X$ is said to be ΔI-convergent to $l\in X$ with respect to the probabilistic norm ν if for each $\epsilon>0$ and $t>0$

$$\{k\in\mathbb{N}:\nu_{\Delta x_k-l}(t)\le 1-\epsilon\}\in I.$$

We write $I_{\nu\Delta}$-$\lim x=l$ and the space of all such sequences as $\Delta(c^{I_\nu})$.

Theorem 2.1. *Let $(X,\nu,*)$ be a probabilistic normed space. Then the following conditions are equivalent:*

(a) $I_{\nu\Delta}$-$\lim x=l$.

(b) $\{k\in\mathbb{N}:\nu_{\Delta x_k-l}(t)\le 1-\epsilon\}\in I_\nu$ *for each* $\epsilon>0$ *and* $t>0$.

(c) $\{k\in\mathbb{N}:\nu_{\Delta x_k-l}(t)>1-\epsilon\}\in\mathscr{F}(I_\nu)$ *for each* $\epsilon>0$ *and* $t>0$.

(d) I-$\lim \nu_{\Delta x_k-l}(t)=1$.

Proof. The proof follows from Definition 2.1. □

Theorem 2.2. *Let $(X,\nu,*)$ be a PNS. If a sequence $x=(x_k)$ is ΔI-convergent then $I_{\nu\Delta}$-limit is unique.*

Proof. Suppose there are two limits l_1 and l_2 with $l_1\ne l_2$. Given $r>0$ such that $(1-r)*(1-r)\ge 1-\epsilon$ for given $\epsilon>0$. Define the sets for $t>0$ as follows:

$$A_{\nu,1}(r,t)=\{k\in\mathbb{N}:\nu_{\Delta x_k-l_1}(t)\le 1-r\},$$

$$B_{\nu,2}(r,t) = \{k \in \mathbb{N} : \nu_{\Delta x_k - l_2}(t) \le 1-r\}.$$

Then by definition of ΔI-convergence $A_{\nu,1}(r,t)$ and $B_{\nu,2}(r,t) \in I$ and hence $C_\nu(r,t) = A_{\nu,1}(r,t) \cup B_{\nu,2}(r,t) \in I$. This implies $C_\nu(r,t)^c \in \mathscr{F}(I)$ so is non-empty. Let $n \in C_\nu(r,t)^c$ then $n \in A_{\nu,1}(r,t)^c \cap B_{\nu,2}(r,t)^c$. So,

$$\nu_{l_1-l_2}(t) \ge \nu_{\Delta x_n - l_1}\left(\frac{t}{2}\right) * \nu_{\Delta x_n - l_2}\left(\frac{t}{2}\right) > (1-r)*(1-r) \ge 1-\epsilon.$$

It follows that $\nu_{l_1-l_2} > 1-\epsilon$. Since ϵ is arbitrary, we have $\nu_{l_1-l_2}(t) = 1 \Rightarrow l_1 = l_2$. This completes the proof. □

Theorem 2.3. *Let $(X,\nu,*)$ be a probabilistic normed space. Then*

(a) *If ν_Δ-$\lim x = l$, then $I_{\nu\Delta}$-$\lim x = l$.*

(b) *If $I_{\nu\Delta}$-$\lim x = l_1$ and $I_{\nu\Delta}$-$\lim y = l_2$, then $I_{\nu\Delta}$-$\lim(x+y) = l_1 + l_2$.*

(c) *If $I_{\nu\Delta}$-$\lim x = l$, then $I_{\nu\Delta}-\alpha x = \alpha l$.*

Proof. (a): Suppose ν_Δ-$\lim x = l$, then by definition for each $\epsilon > 0$ and $t > 0$ there exists $N > 0$ such that

$$\nu_{\Delta x_k - l}(t) > 1-\epsilon \quad \text{for each } k > N.$$

Observe that $A(t) = \{k \in \mathbb{N} : \nu_{\Delta x_k - l}(t) \le 1-\epsilon\} \subseteq \{1,2,3,\cdots,N-1\}$. Since I is admissible ideal, therefore $A(t) \in I$. Hence $I_{\nu\Delta}$-$\lim x = l$.

(b): Let $I_{\nu\Delta}$-$\lim x = l_1$ and $I_{\nu\Delta}$-$\lim y = l_2$. For given $\epsilon > 0$ and $t > 0$ given $r > 0$ with $(1-r)*(1-r) > 1-\epsilon$. By definition the sets

$$A_{\nu,1}(r,t) = \{k \in \mathbb{N} : \nu_{\Delta x_k - l_1}(t) \le 1-r\} \in I,$$

$$B_{\nu,2}(r,t) = \{k \in \mathbb{N} : \nu_{\Delta y_k - l_2}(t) \le 1-r\} \in I.$$

$C_\nu(r,t) = A_{\nu,1}(r,t) \cup B_{\nu,2}(r,t) \in I$ so that $C_\nu(r,t)^c \in \mathscr{F}(I)$. We prove

$$C_\nu(r,t)^c \subseteq \{k \in \mathbb{N} : \nu_{(\Delta x_k + \Delta y_k)-(l_1+l_2)}(t) > 1-\epsilon\}.$$

Let $k \in C_\nu(r,t)^c$, then

$$\nu_{(\Delta x_k + \Delta y_k)-(l_1+l_2)}(t) \ge \nu_{\Delta x_k - l_1}\left(\frac{t}{2}\right) * \nu_{\Delta y_k - l_2}\left(\frac{t}{2}\right) > (1-r)*(1-r) > 1-\epsilon.$$

Therefore,

$$C_\nu(r,t)^c \subseteq \{k \in \mathbb{N} : \nu_{(\Delta x_k + \Delta y_k)-(l_1+l_2)}(t) > 1-\epsilon\}.$$

Hence

$$\{k \in \mathbb{N} : \nu_{(\Delta x_k + \Delta y_k)-(l_1+l_2)}(t) > 1-\epsilon\} \in I.$$

Thus $I_{\nu\Delta}$-$\lim(x_k + y_k) = l_1 + l_2$.

(c): The proof holds for $\alpha = 0$. Let $\alpha \ne 0$. We are given $I_{\nu\Delta}$-$\lim x = l$, therefore the set

$$A(t) = \{k \in \mathbb{N} : \nu_{\Delta x_k - l}(t) > 1-\epsilon\} \in \mathscr{F}(I).$$

We prove $A(t) \subseteq \{k \in \mathbb{N} : \nu_{\Delta \alpha x_k - \alpha l}(t) > 1-\epsilon\}$. Let $k \in A(t)$. Then by definition

$$\nu_{\Delta x_k - l}(t) > 1-\epsilon.$$

Now

$$\nu_{\Delta\alpha x_k-\alpha l}(t)=\nu_{\Delta x_k-l}\left(\frac{t}{|\alpha|}\right)\geq \nu_{\Delta x_k-l}(t)*\nu_0\left(\frac{t}{|\alpha|}-t\right)=\nu_{\Delta x_k-l}(t)*1=\nu_{\Delta x_k-l}(t)>1-\epsilon.$$

Hence, we have $A(t)\subseteq\{k\in\mathbb{N}:\nu_{\Delta\alpha x_k-\alpha l}(t)>1-\epsilon\}$ and therefore $I_{\nu\Delta}\text{-}\lim \alpha x=\alpha l$. □

2.2 ΔI^*-Convergence in PNS

In this section, we introduce the concept of ΔI^*-convergence of sequences in probabilistic normed space.

Definition 2.2. Consider PNS $(X,\nu,*)$. A sequence $x=(x_k)\in X$ is said to be ΔI^*-convergent to $l\in X$ with respect to the probabilistic norm ν if there exists $K=\{k_m:k_1<k_2<\cdots\}\subseteq\mathbb{N}$ such that $K\in\mathscr{F}(I)$ and $\nu_\Delta\text{-}\lim_m x_{k_m}=l$. We write $I^*_{\nu\Delta}\text{-}\lim x=l$.

Theorem 2.4. *Let $(X,\nu,*)$ be a PNS and I be an admissible ideal. If $I^*_{\nu\Delta}\text{-}\lim x=l$ then $I_{\nu\Delta}$-$\lim x=l$.*

Proof. Let $I^*_{\nu\Delta}\text{-}\lim x=l$. Then by definition there exists $K=\{k_m:k_1<k_2\cdots\}\in\mathscr{F}(I)$ ($K^c=H$(say)$\in I$)and $\nu_\Delta\text{-}\lim_m x_{k_m}=l$. Then for each $\epsilon>0$ and $t>0$ there exists $N>0$ such that $\nu_{\Delta x_{k_m}-l}(t)>1-\epsilon$ for all $m>N$. Since $\{k_m\in K:\nu_{\Delta x_{k_m}-l}(t)\leq 1-\epsilon\}\subseteq\{k_1<k_2<\cdots<k_{N-1}\}$ and I is an admissible ideal, we have

$$\{k_m\in K:\nu_{\Delta x_{k_m}-l}(t)\leq 1-\epsilon\}\in I.$$

Hence

$$\{k\in\mathbb{N}:\nu_{\Delta x_k-l}(t)\leq 1-\epsilon\}\subseteq H\cup\{k_1<k_2<\cdots k_{N-1}\}\in I$$

for each $\epsilon>$ and $t>0$. It follows $I_{\nu\Delta}\text{-}\lim x=l$ □

Remark 2.1. The converse of above theorem is not necessarily true which is shown by the below given example.

Example 2.1. Consider the normed space $(\mathbb{R},|\cdot|)$ with the usual norm and let $a*b=ab$ for all $a,b\in[0,1]$. Define

$$\nu_x(t):=\frac{t}{t+|x|}\quad\text{for all } x\in\mathbb{R} \text{ and every } t>0.$$

Then $(\mathbb{R},\nu,*)$ is a PNS. Let $\mathbb{N}=\bigcup_j D_j$ be a decomposition of $\mathbb{N}$ such that for any $n\in\mathbb{N}$ each D_j contains infinitely many j's where $j\geq n$ and $D_j\cap D_n=\phi$. Let I be the class of all subsets of $\mathbb{N}$ which intersects with at most a finite number of D_j's, then I is an admissible ideal. Define a sequence $x_n=\frac{1}{j}$ if $n\in D_j$. Then

$$\nu_{\Delta x_n}(t)=\frac{t}{t+|\Delta x_n|}\to 1\ \text{ as }\ n\to\infty.$$

Hence $I_{\nu\Delta}\text{-}\lim x_n=0$. Now suppose that $I^*_{\nu\Delta}\text{-}\lim x_n=0$, then there exists $K=\{n_j:n_1<n_2<\cdots\}\subseteq\mathbb{N}$ with $K\in\mathscr{F}(I)$ and $\nu_\Delta\text{-}\lim x_{n_j}=0$. Since $K\in\mathscr{F}(I)$, we have $K^c=H(\text{say})\in I$. Then, there exists $p\in\mathbb{N}$ such that $H\subset\left(\bigcup_{n=1}^{p}D_n\right)$.

But then $D_{p+1} \subset K$ and therefore

$$x_{n_j} = \frac{1}{p+1} > 0$$

for infinitely many n_j's from K which contradicts ν_Δ-$\lim x_{n_j} = 0$. Hence we get a contradiction.

Theorem 2.5. *Let $(X, \nu, *)$ be a PNS and I satisfies AP condition. Then $I_{\nu\Delta}$-$\lim x = l$ implies $I^*_{\nu\Delta}$-$\lim x = l$.*

Proof. Suppose I be an admissible ideal that satisfies AP condition and $I_{\nu\Delta}$-$\lim x = \xi$. By definition for each $\epsilon > 0$ and $t > 0$, we have $\{k \in \mathbb{N} : \nu_{\Delta x_k - \xi}(t) \le 1 - \epsilon\} \in I$. Define the set A_p for $p \in \mathbb{N}$

$$A_p = \left\{k \in \mathbb{N} : 1 - \frac{1}{p} \le \nu_{\Delta x_k - \xi}(t) < 1 - \frac{1}{p+1}\right\}.$$

Observe that $\{A_1, A_2 \cdots\}$ is a countable set, belongs to I and $A_i \cap A_j = \phi$ for $i \ne j$. There exists a countable family of sets $\{B_1, B_2 \cdots\} \in I$ such that the symmetric difference $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in I$. Hence $B^c = K(\text{say}) \in \mathscr{F}(I)$. We will prove $(x_k)_{k \in K}$ is $\Delta\nu$-convergent to ξ. Let $\eta > 0$ and $t > 0$, choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \eta$. Then

$$\{k \in \mathbb{N} : \nu_{\Delta x_k - \xi}(t) \le 1 - \eta\} \subset \left\{k \in \mathbb{N} : \nu_{\Delta x_k - \xi}(t) \le 1 - \frac{1}{q}\right\} \subset \bigcup_{i=1}^{q+1} A_i.$$

Since $A_i \Delta B_i$, $i = 1, 2, \cdots, q+1$ are finite, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{q+1} B_i\right) \cap \{k : k \ge k_0\} = \left(\bigcup_{i=1}^{q+1} A_i\right) \cap \{k : k \ge k_0\}.$$

If $k \ge k_0$ and $k \in K$ then $k \notin \bigcup_{i=1}^{q+1} B_i$. Therefore, $k \notin \bigcup_{i=1}^{q+1} A_i$. Hence for every $k \ge k_0$ and $k \in K$, we have $\nu_{\Delta x_k - \xi}(t) > 1 - \eta$.

Since $\eta > 0$ is arbitrary, we have $I^*_{\nu\Delta}$-$\lim x = \xi$. □

Theorem 2.6. *Let $(X, \nu, *)$ be a PNS. Then the following statements are equivalent:*

(i) $I^*_{\nu\Delta}$-$\lim x = l$.

(ii) *There exist two sequences $y = (y_k)$ and $z = (z_k)$ in X such that $x = y + z$, ν_Δ-$\lim y = \xi$ and the set $\{k : z_k \ne \theta\} \in I$, where θ denotes the zero element of X.*

Proof. Suppose (i) holds. Then there exists $K = \{k_m : k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ such that $K \in \mathscr{F}(I)$ and ν_Δ-$\lim x_{k_m} = l$. Define the sequences y and z as follows:

$$y_k = \begin{cases} x_k & \text{if } k \in K \\ l & \text{if } k \in K^c \end{cases}$$

and $z_k = x_k - y_k$ for all $k \in \mathbb{N}$. For each $\epsilon > 0$, $t > 0$ and $k \in K^c$, we have $\nu_{\Delta y_k - l}(t) = 1 > 1 - \epsilon$. Thus ν_Δ-$\lim y = l$. Since $\{k : z_k \ne \theta\} \subset K^c$, we have $\{k : z_k \ne \theta\} \in I$.

Let (ii) holds. Then the set $K = \{k : z_k \ne \theta\} \in \mathscr{F}(I)$ is an infinite set. Let $K = \{k_m : k_1 < k_2 < \cdots\}$. Since $x_{k_m} = y_{k_m}$ and ν_Δ-$\lim y = l$, ν_Δ-$\lim x_{k_m} = l$. Therefore $I^*_{\nu\Delta}$-$\lim x = l$. □

3. ΔI-Limit Points and ΔI-Cluster Points in PNS

Definition 3.1. Let $(X,\nu,*)$ be a probabilistic normed space, $l \in X$ is said to be Δ-limit point of sequence $x=(x_k)\in X$ with respect to the probabilistic norm ν if there is a subsequence of x that Δ-converges to l with respect to the probabilistic norm ν. By $\mathscr{L}_{\nu\Delta}(x)$, we denote the set of all Δ-limit points of the sequence x.

Definition 3.2. Let $(X,\nu,*)$ be a PNS. An element $l\in X$ is said to be ΔI-limit point of the sequence $x=(x_k)\in X$ with respect to the probabilistic norm $\nu(I_{\nu\Delta}$-limit point) if there is a subset $K=\{k_1<k_2<\cdots\}$ of $\mathbb{N}$ such that $K\in I$ and ν_Δ-$\lim x_{k_m}=l$. We denote by $\Lambda^I_{\nu\Delta}(x)$, the set of all $I_{\nu\Delta}$-limit points of the sequence $x=(x_k)$.

Definition 3.3. Let $(X,\nu,*)$ be a PNS. An element $l\in X$ is said to be ΔI-cluster point of $x=(x_k)\in X$ with respect to the probabilistic norm ν if for each $\epsilon>0$ and $t>0$

$$K=\{k\in\mathbb{N}:\nu_{\Delta x_k-l}(t)>1-\epsilon\}\notin I.$$

We denote by $\Gamma^I_{\nu\Delta}(x)$ the set of all $I_{\nu\Delta}$-cluster points of the sequence x.

Theorem 3.1. *Let $(X,\nu,*)$ be a PNS. Then $\Lambda^I_{\nu\Delta}(x)\subset\Gamma^I_{\nu\Delta}(x)\subset\mathscr{L}_{\nu\Delta}(x)$, where $x=(x_k)\in X$.*

Proof. Let $l\in\Lambda^I_{\nu\Delta}(x)$, there exists $K=\{k_m:k_1<k_2<\cdots\}\subseteq\mathbb{N}$ such that $K\notin I$ and ν_Δ-$\lim x_{k_m}=l$. For each $\epsilon>0$ and $t>0$, there exists $N\in\mathbb{N}$ such that for $k>N$, we have $\nu_{\Delta x_k-l}(t)>1-\epsilon$. So,

$$\{k\in\mathbb{N}:\nu_{\Delta x_k-l}(t)>1-\epsilon\}\supset\{k_{N+1},k_{N+2},\cdots\}$$

and thus $\{k\in\mathbb{N}:\nu_{\Delta x_k-l}(t)>1-\epsilon\}\notin I$ which implies $l\in\Gamma^I_{\nu\Delta}(x)$.
Let $l\in\Gamma^I_{\nu\Delta}(x)$ then for each $\epsilon>0$ and $t>0$, we have $\{k\in\mathbb{N}:\nu_{\Delta x_k-l}(t)>1-\epsilon\}\notin I$. Let $K=\{k_m:k_1<k_2<\cdots\}$. Then there is a subsequence (x_{k_n}) of (x_n) that Δ-converges to l with respect to the probabilistic norm ν. Hence $l\in\mathscr{L}_{\nu\Delta}(x)$. □

Theorem 3.2. *Let $(X,\nu,*)$ be a PNS and $x=(x_k)$ be a sequence in X. Then $\Lambda^I_{\nu\Delta}(x)=\Gamma^I_{\nu\Delta}(x)=\{l\}$, if $I_{\nu\Delta}$-$\lim x=l$.*

Proof. Let $l_1,l_2\in\Lambda^I_{\nu\Delta}(x)$, with $l_1\neq l_2$. Then there exist two subsets K,K', $K=\{k_m:k_1<k_2\cdots\}$ and $K'=\{p_m:p_1<p_2<\cdots\}$ of $\mathbb{N}$ such that

$$K\notin I \text{ and } \nu_\Delta\text{-}\lim x_{k_m}=l_1,$$

$$K'\notin I \text{ and } \nu_\Delta\text{-}\lim x_{p_m}=l_2.$$

Given $\epsilon>0$ and $t>0$ there exists $N\in\mathbb{N}$ with $m>N$, we have $\nu_{\Delta x_{p_m}-l_2}(t)>1-\epsilon$. Hence

$$A=\{p_m\in K':\nu_{\Delta x_{p_m}-l_2}(t)\le 1-\epsilon\}\subset\{p_m:p_1<p_2<\cdots p_N\}.$$

As I is an admissible ideal so $A\in I$. If $B=\{p_m\in K':\nu_{\Delta p_m-\eta}(t)>1-\epsilon\}$,then $B\notin I$. Otherwise if $B\in I$, then $A\cup B=K'\in I$ which is a contradiction. Also $I_{\nu\Delta}$-$\lim x=l_1$, we have

$$C=\{k\in\mathbb{N}:\nu_{\Delta x_k-l_1}(t)\le 1-\epsilon\}\in I.$$

Hence

$$C^c=\{k\in\mathbb{N}:\nu_{\Delta x_k-l_1}(t)>1-\epsilon\}\in\mathscr{F}(I).$$

Since for every $l_1 \neq l_2$, $B \cap C^c = \phi$, $B \subset C$. Hence $\Lambda^I_{\nu\Delta}(x) = l_1$. On the other hand, suppose $l_1, l_2 \in \Gamma^I_{\nu\Delta}(x)$, with $l_1 \neq l_2$. Then for each $\epsilon > 0$ and $t > 0$, we have

$$A = \{k \in \mathbb{N} : \nu_{\Delta x_k - l_1}(t) > 1 - \epsilon\} \notin I,$$

$$B = \{k \in \mathbb{N} : \nu_{\Delta x_k - l_2}(t) > 1 - \epsilon\} \notin I.$$

Observe that $A \cap B = \phi$ and therefore $B \subset A^c$. Also, $I_{\nu\Delta}$-$\lim x = l_1$ implies $A^c = \{k \in \mathbb{N} : \nu_{\Delta x_k - l_1}(t) \leq 1 - \epsilon\} \in I$. Hence $B \in I$, which is a contradiction. Therefore, $\Gamma^I_{\nu\Delta}(x) = l_1$. □

Theorem 3.3. *Let $(X, \nu, *)$ be a PNS. If $x = (x_k)$ and $y = (y_k)$ be two sequences in X and $A = \{k \in \mathbb{N} : x_k \neq y_k\} \in I$. Then $\Lambda^I_{\nu\Delta}(x) = \Lambda^I_{\nu\Delta}(y)$ and $\Gamma^I_{\nu\Delta}(x) = \Gamma^I_{\nu\Delta}(y)$.*

Proof. Suppose $l \in \Lambda^I_{\nu\Delta}(x)$, then by definition there exists $K = \{k_m : k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ such that $K \notin I$ and ν_Δ-$\lim x_{k_m} = l$. For each $\epsilon > 0$ and $t > 0$, we can find $N \in \mathbb{N}$ such that $\nu_{\Delta x_{k_m} - l}(t) > 1 - \epsilon$ for $m > N$. Define $K_1 = K \cap A$ and $K_2 = K - A$. $A \in I$ implies $K_1 \in I$. As $K = K_1 \cup K_2$ and $K \notin I$ so $K_2 \notin I$. It is obvious that the subsequence $(y_k)_{k \in K_2}$ of the sequence $y = (y_k)$ is $\Delta\nu$-convergent to l. Hence $l \in \Lambda^I_{\nu\Delta}(y)$ and therefore $\Lambda^I_{\nu\Delta}(x) \subseteq \Lambda^I_{\nu\Delta}(y)$. Similarly, we can prove $\Lambda^I_{\nu\Delta}(y) \subseteq \Lambda^I_{\nu\Delta}(x)$. Thus $\Lambda^I_{\nu\Delta}(x) = \Lambda^I_{\nu\Delta}(y)$. Let $l \in \Gamma^I_{\nu\Delta}(x)$, then for each $\epsilon > 0$ and $t > 0$, we have

$$B = \{k \in \mathbb{N} : \nu_{\Delta x_k - l}(t) > 1 - \epsilon\} \notin I.$$

Define $C = \{k \in \mathbb{N} : \nu_{\Delta y_k - l}(t) > 1 - \epsilon\}$. We need to show $C \notin I$. Suppose on contrary $C \in I$ then $C^c \in \mathscr{F}(I)$. Also we have $A_c \in \mathscr{F}(I)$. Thus $C_c \cap A_c \in \mathscr{F}(I)$. $C^c \cap A^c \subset B^c$ implies $B^c \in \mathscr{F}(I)$. Hence $B \in I$ which is a contradiction. So $C \notin I$ and $\Gamma^I_{\nu\Delta}(x) \subseteq \Gamma^I_{\nu\Delta}(y)$. Similarly $\Gamma^I_{\nu\Delta}(y) \subseteq \Gamma^I_{\nu\Delta}(x)$ and hence $\Gamma^I_{\nu\Delta}(x) = \Gamma^I_{\nu\Delta}(y)$. □

4. Conclusion

In the present paper we have introduced and studied the notion of difference I-convergence of sequence in PNS and elementary properties of this convergence. We investigated the general type of I-convergence for difference sequences, that is, Difference Ideal Convergence in Probabilistic Normed Spaces in more general setting. These definitions and results provide new tools to deal with the convergence problems of sequences occurring in many branches of science and engineering.

Acknowledgements

The authors thank the following people for their help and support at different stages of this work:

(1) Prof. Bernardo Lafuerza-Guillén, Department of Mathematics, University of Almería, 04120 Almería, Spain.

(2) Prof. Mohammad Imdad, Chairman, Department of Mathematics, Aligarh Muslim University, Aligarh, India.

Funding

This work is financially supported by Mohd. Faisal Khan, Assistant Professor in College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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