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Research Article

Hermite-Hadamard Type Inequalities via the Montgomery Identity

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Abstract. The main aim of this manuscript is to prove the result for Hermite-Hadamard types inequalities and to strengthen our results by giving applications for means. The proof of the result is based on the Montgomery identity. We use the Montgomery identity to establish a new identity regarding the Hermite-Hadamard inequality. Based on this identity with a class of convex and monotone functions and Jensen's inequality, we obtain various results for the inequality. At the end, we also present applications for special bivariate means.

Keywords. Montgomery identity; Convex function; Hermite-Hadamard inequality; Means

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1. Introduction

Convex sets and convex functions play a vital role in the field of pure and applied sciences. In the literature many inequalities have been proposed for different classes of convex functions such as pre-invex, s-convex, quasi-convex and GG-convex etc. Among various types of inequalities introduced in the area of convexity, Hermite-Hadamard inequality is of much interest. The Hermite-Hadamard double inequality is the first fundamental result for convex functions with a natural geometrical interpretation and a loose number of applications for particular inequalities. The statement of this inequality is (see [16]):

Let I be an interval in $\mathbf{R}$. Then a function $h: I \to \mathbf{R}$ is a convex function if the following inequality holds

$$h\left(\frac{a_1+a_2}{2}\right) \le \frac{1}{a_2-a_1}\int_{a_1}^{a_2} h(x)dx \le \frac{h(a_1)+h(a_2)}{2}, \tag{1}$$

where $a_1, a_2 \in I$ with $a_1 < a_2$. If both the inequalities in (1) hold in the reverse direction the function h is concave on I.

For more related results, generalizations, improvements, variants, refinements and applications to Hermite-Hadamard inequality (see [3–7, 9–15, 17–29, 31, 32, 34, 37]).

In [2] Anderson proved the following Montgomery Identity using the conformal fractional integrals:

Lemma 1. *Let $a_1, a_2, s, t \in \mathbf{R}$ and let $h: [a_1, a_2] \to \mathbf{R}$ be a differentiable function. Then for $\alpha \in (0,1]$, we have*

$$h(t) = \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(s)s^{\alpha-1}ds + \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} p(t,s)h'(s)ds, \tag{2}$$

$$p(t,s) = \begin{cases} \frac{s^\alpha - a_1^\alpha}{\alpha}, & a_1 \le s < t; \\ \frac{s^\alpha - a_2^\alpha}{\alpha}, & t \le s \le a_2. \end{cases}$$

Let $h: [a_1, a_2] \to \mathbf{R}$ be an integrable function. Then we define $\Delta(h; a_1, a_2)$ as:

$$\Delta(h; a_1, a_2) := \frac{1}{a_2-a_1}\int_{a_1}^{a_2} h(x)dx - h\left(\frac{a_1+a_2}{2}\right). \tag{3}$$

Many researchers have shown their extensive attention on generalizations, extensions and variants of Hermite-Hadamard inequality. The main aim of this manuscript is to prove the result for Hermite-Hadamard types inequalities and to strengthen our results by giving applications for means. The proof of the result is based on the Montgomery identity. We use the Montgomery identity to establish a new identity regarding Hermite-Hadamard inequality. Based on this identity with a class of convex and monotone functions and Jensen's inequality, we obtain various new results for the inequality. At the end, we also present applications for special bivariate means.

In Section 2 we establish several Hermite-Hadamard type inequalities by using Montgomery identity. In Section 3, we present applications of the main results for different means.

2. Main Results

We begin this section with the following lemma, which will help in the proving of our results:

Lemma 2. *Let $a_1, a_2 \in \mathbf{R}^+$ and $h: [a_1, a_2] \to \mathbf{R}$ be a differentiable function. Then for $\alpha \in (0,1]$, the following identity holds:*

$$\Delta(h; a_1, a_2) = \frac{1}{(a_2^\alpha - a_1^\alpha)(a_2-a_1)}\left[\int_{a_1}^{a_2} s^\alpha (a_2-a_1)h'(s)ds - \int_{a_1}^{a_2} a_2^\alpha (s-a_1)h'(s)ds\right.$$

$$-\int_{a_1}^{a_2} a_1^{\alpha}(a_2-s)h'(s)ds\Big] - \frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^{\alpha}-a_1^{\alpha})h'(s)ds$$
$$-\frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{\frac{a_1+a_2}{2}}^{a_2}(s^{\alpha}-a_2^{\alpha})h'(s)ds.$$

Proof. If we put $t=\frac{a_1+a_2}{2}$ in (2), we get

$$\begin{aligned} h\left(\frac{a_1+a_2}{2}\right) &= \frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2} h(s)s^{\alpha-1}ds + \frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2} p\left(\frac{a_1+a_2}{2},s\right)h'(s)ds \\ &= \frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2} h(s)s^{\alpha-1}ds + \frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^{\alpha}-a_1^{\alpha})h'(s)ds \\ &\quad + \frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{\frac{a_1+a_2}{2}}^{a_2}(s^{\alpha}-a_2^{\alpha})h'(s)ds. \end{aligned} \tag{4}$$

We integrate (2) with respect to t and then multiplying by $\frac{1}{a_2-a_1}$, we get

$$\begin{aligned} \frac{1}{a_2-a_1}\int_{a_1}^{a_2} h(t)dt &= \frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\int_{a_1}^{a_2} h(s)s^{\alpha-1}dsdt \\ &\quad + \frac{1}{a_2-a_1}\frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2}\int_{a_1}^{a_2} p(t,s)h'(s)dsdt\Big] \\ &= \frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2} h(s)s^{\alpha-1}ds + \frac{\alpha}{(a_2^{\alpha}-a_1^{\alpha})(a_2-a_1)}\Big[\int_{a_1}^{a_2}\int_{a_1}^{s}\frac{s^{\alpha}-a_2^{\alpha}}{\alpha}h'(s)dtds \\ &\quad + \int_{a_1}^{a_2}\int_{s}^{a_2}\frac{s^{\alpha}-a_1^{\alpha}}{\alpha}h'(s)dtds\Big] \\ &= \frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2} h(s)s^{\alpha-1}ds + \frac{1}{(a_2^{\alpha}-a_1^{\alpha})(a_2-a_1)}\Big[\int_{a_1}^{a_2}(s^{\alpha}-a_2^{\alpha})(s-a_1)h'(s)ds \\ &\quad + \int_{a_1}^{a_2}(s^{\alpha}-a_1^{\alpha})(a_2-s)h'(s)ds\Big] \\ &= \frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2} h(s)s^{\alpha-1}ds + \frac{1}{(a_2^{\alpha}-a_1^{\alpha})(a_2-a_1)}\Big[\int_{a_1}^{a_2} s^{\alpha}(a_2-a_1)h'(s)ds \\ &\quad - \int_{a_1}^{a_2} a_2^{\alpha}(s-a_1)h'(s)ds - \int_{a_1}^{a_2} a_1^{\alpha}(a_2-s)h'(s)ds\Big] \end{aligned} \tag{5}$$

Now, from (4) and (5), we have

$$\begin{aligned} \Delta(h;a_1,a_2) &= \frac{1}{(a_2^{\alpha}-a_1^{\alpha})(a_2-a_1)}\Big[\int_{a_1}^{a_2} s^{\alpha}(a_2-a_1)h'(s)ds - \int_{a_1}^{a_2} a_2^{\alpha}(s-a_1)h'(s)ds \\ &\quad - \int_{a_1}^{a_2} a_1^{\alpha}(a_2-s)h'(s)ds\Big] - \frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^{\alpha}-a_1^{\alpha})h'(s)ds \\ &\quad - \frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{\frac{a_1+a_2}{2}}^{a_2}(s^{\alpha}-a_2^{\alpha})h'(s)ds. \end{aligned}$$

□

In the following theorem we obtain inequalities for left difference of Hermite-Hadamard inequality by taking h' decreasing.

Theorem 1. *Let $a_1,a_2 \in \mathbf{R}^+$ and $h:[a_1,a_2]\to\mathbf{R}$ be a decreasing differentiable function. Then for $\alpha\in(0,1]$, the following inequalities hold:*

$$\Delta(h;a_1,a_2) \ge \frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\left[\int_{a_1}^{a_2} s^\alpha(a_2-a_1)h'(s)ds - \int_{a_1}^{a_2} a_1^\alpha(a_2-s)h'(s)ds\right] - \frac{1}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^\alpha-a_1^\alpha)h'(s)ds - \frac{1}{a_2^\alpha-a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2}(s^\alpha-a_2^\alpha)h'(s)ds.$$

$$\Delta(h;a_1,a_2) \ge \frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\left[\int_{a_1}^{a_2} s^\alpha(a_2-a_1)h'(s)ds - \int_{a_1}^{a_2} a_2^\alpha(s-a_1)h'(s)ds\right] - \frac{1}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^\alpha-a_1^\alpha)h'(s)ds - \frac{1}{a_2^\alpha-a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2}(s^\alpha-a_2^\alpha)h'(s)ds.$$

$$\Delta(h;a_1,a_2) \ge \frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\left[\int_{a_1}^{a_2} s^\alpha(a_2-a_1)h'(s)ds - \int_{a_1}^{a_2} a_2^\alpha(s-a_1)h'(s)ds - \int_{a_1}^{a_2} a_1^\alpha(a_2-s)h'(s)ds\right] - \frac{1}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^\alpha-a_1^\alpha)h'(s)ds.$$

Proof. Using Lemma 2, we have

$$\Delta(h;a_1,a_2) = \frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\left[\int_{a_1}^{a_2} s^\alpha(a_2-a_1)h'(s)ds - \int_{a_1}^{a_2} a_2^\alpha(s-a_1)h'(s)ds - \int_{a_1}^{a_2} a_1^\alpha(a_2-s)h'(s)ds\right] - \frac{1}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^\alpha-a_1^\alpha)h'(s)ds - \frac{1}{a_2^\alpha-a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2}(s^\alpha-a_2^\alpha)h'(s)ds. \tag{6}$$

Since h is decreasing, so we have

$$-\int_{a_1}^{a_2} a_2^\alpha(s-a_1)h'(s)ds \ge 0. \tag{7}$$

By using (7) in (6), we obtain

$$\Delta(h;a_1,a_2) \ge \frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\left[\int_{a_1}^{a_2} s^\alpha(a_2-a_1)h'(s)ds - \int_{a_1}^{a_2} a_1^\alpha(a_2-s)h'(s)ds\right] - \frac{1}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^\alpha-a_1^\alpha)h'(s)ds - \frac{1}{a_2^\alpha-a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2}(s^\alpha-a_2^\alpha)h'(s)ds.$$

Similarly, we can prove the other inequalities. □

In the next theorem we obtain bounds for the left difference of Hermite-Hadamard inequality by using the convexity of $|h'|$.

Theorem 2. *Let $a_1,a_2 \in \mathbf{R}^+$ and $h:[a_1,a_2]\to\mathbf{R}$ be a differentiable function such that $|h'|$ is convex. Then for $\alpha\in(0,1]$, the following inequalities hold:*

$$|\Delta(h;a_1,a_2)| \le \frac{a_2^\alpha(a_2-a_1)}{(a_2^\alpha-a_1^\alpha)}\left\{\frac{|h'(a_1)|+2|h'(a_2)|}{6}\right\} + \frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\left[\int_{a_1}^{a_2} s^\alpha(a_2-a_1)|h'(s)|ds\right.$$

$$+\int_{a_1}^{a_2} a_1^\alpha(a_2-s)|h'(s)|ds\Bigg]+\frac{1}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^\alpha-a_1^\alpha)|h'(s)|ds$$
$$+\frac{1}{a_2^\alpha-a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2^\alpha-s^\alpha)|h'(s)|ds. \tag{8}$$

$$|\Delta(h;a_1,a_2)|\le\frac{a_2-a_1}{a_2^\alpha-a_1^\alpha}\left\{\frac{a_1^\alpha|h'(a_1)|+a_2^\alpha|h'(a_2)|}{4}+a_1^{\alpha-1}a_2\left\{\frac{|h'(a_1)|+|h'(a_2)|}{12}\right\}+a_1a_2^{\alpha-1}\right.$$
$$\left.\times\left\{\frac{|h'(a_1)|+|h'(a_2)|}{12}\right\}+\frac{a_1^\alpha|h'(a_2)|+a_2^\alpha|h'(a_1)|}{12}\right\}+\frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}$$
$$\cdot\left[\int_{a_1}^{a_2}a_2^\alpha(s-a_1)|h'(s)|ds+\int_{a_1}^{a_2}a_1^\alpha(a_2-s)|h'(s)|ds\right]$$
$$+\frac{1}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^\alpha-a_1^\alpha)|h'(s)|ds+\frac{1}{a_2^\alpha-a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2^\alpha-s^\alpha)|h'(s)|ds. \tag{9}$$

$$|\Delta(h;a_1,a_2)|\le\frac{(a_2-a_1)a_1^\alpha}{a_2^\alpha-a_1^\alpha}\left\{\frac{2|h'(a_1)|+|h'(a_2)|}{6}\right\}+\frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\Bigg[\int_{a_1}^{a_2}a_2^\alpha(a_1-s)h'(s)ds$$
$$+\int_{a_1}^{a_2}s^\alpha(a_2-a_1)|h'(s)|ds\Bigg]+\frac{1}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^\alpha-a_1^\alpha)|h'(s)|ds$$
$$+\frac{1}{a_2^\alpha-a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2^\alpha-s^\alpha)|h'(s)|ds. \tag{10}$$

$$|\Delta(h;a_1,a_2)|\le\frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\Bigg[\int_{a_1}^{a_2}a_2^\alpha(s-a_1)|h'(s)|ds+\int_{a_1}^{a_2}s^\alpha(a_2-a_1)|h'(s)|ds$$
$$+\int_{a_1}^{a_2}a^\alpha(a_2-s)|h'(s)|ds\Bigg]$$
$$+\frac{a_2-a_1}{a_2^\alpha-a_1^\alpha}\Bigg[\frac{15a_1^\alpha|h'(a_1)|+a_2^\alpha|h'(a_2)|}{64}+a_1^{\alpha-1}a_2\left\{\frac{11|h'(a_1)|+5|h'(a_2)|}{192}\right\}$$
$$+a_1a_2^{\alpha-1}\left\{\frac{11|h'(a_1)|+5|h'(a_2)|}{192}\right\}+\frac{11a_1^\alpha|h'(a_2)|+5a_2^\alpha|h'(a_1)|}{192}$$
$$-\frac{3a_1^\alpha|h'(a_2)|+a_1^\alpha|h'(a_1)|}{8}\Bigg]+\frac{1}{a_2^\alpha-a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2^\alpha-s^\alpha)|h'(s)|ds. \tag{11}$$

$$|\Delta(h;a_1,a_2)|\le\frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\Bigg[\int_{a_1}^{a_2}a_2^\alpha(s-a_1)|h'(s)|ds+\int_{a_1}^{a_2}s^\alpha(a_2-a_1)|h'(s)|ds$$
$$+\int_{a_1}^{a_2}a_1^\alpha(a_2-s)|h'(s)|ds\Bigg]+\frac{1}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^\alpha-a_1^\alpha)|h'(s)|ds$$
$$+(a_2-a_1)\left\{\frac{|h'(a_1)|+2|h'(a_2)|}{24}\right\}. \tag{12}$$

Proof. From Lemma 2 and using the property of the modulus, we can write

$$|\Delta(h;a_1,a_2)|\le\frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\Bigg[\int_{a_1}^{a_2}a_2^\alpha(s-a_1)|h'(s)|ds+\int_{a_1}^{a_2}s^\alpha(a_2-a_1)|h'(s)|ds$$

$$+\int_{a_1}^{a_2} a_1^{\alpha}(a_2-s)|h'(s)|ds\Bigg] + \frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^{\alpha}-a_1^{\alpha})|h'(s)|ds$$
$$+\frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2^{\alpha}-s^{\alpha})|h'(s)|ds. \tag{13}$$

Now by change of variable and using the convexity of $|h'|$, we get

$$\int_{a_1}^{a_2}(s-a_1)|h'(s)|ds \le (a_2-a_1)^2\int_0^1((1-\lambda)\lambda|h'(a_1)|+(1-\lambda)^2|h'(a_2)|)d\lambda$$
$$=(a_2-a_1)^2\left[\frac{|h'(a_1)|+2|h'(a_2)|}{6}\right]. \tag{14}$$

By using (14) in (13), we obtain (8).

Now, we prove the inequality (9).

Similarly, as above by change of variable and the convexity of $|h'|$, we have

$$\int_{a_1}^{a_2} s^{\alpha}|h'(s)|ds = (a_2-a_1)\int_0^1 (a_1\lambda+(1-\lambda)a_2)^{\alpha}|h'(a_1\lambda+(1-\lambda)a_2)|d\lambda$$
$$=(a_2-a_1)\int_0^1 (a_1\lambda+(1-\lambda)a_2)^{\alpha-1}(a_1\lambda+(1-\lambda)a_2)|h'(a_1\lambda+(1-\lambda)a_2)|d\lambda$$
$$\le (a_2-a_1)\int_0^1 (a_1^{\alpha-1}\lambda+(1-\lambda)a_2^{\alpha-1})(a_1\lambda+(1-\lambda)a_2)|h'(a_1\lambda+(1-\lambda)a_2)|d\lambda$$
$$\le (a_2-a_1)\int_0^1 (a_1^{\alpha}\lambda^2+a_1^{\alpha-1}\lambda(1-\lambda)a_2+\lambda(1-\lambda)a_1a_2^{\alpha-1}+(1-\lambda)^2a_2^{\alpha})[\lambda|h'(a_1)|$$
$$+(1-\lambda)|h'(a_2)|]d\lambda$$
$$=(a_2-a_1)\Bigg[\frac{1}{4}a_1^{\alpha}|h'(a_1)|+\frac{1}{12}a_1^{\alpha-1}a_2|h'(a_1)|+\frac{1}{12}a_1a_2^{\alpha-1}|h'(a_1)|+\frac{1}{12}a_2^{\alpha}|h'(a_1)|$$
$$+\frac{1}{12}a_1^{\alpha}|h'(a_2)|+\frac{1}{12}a_1^{\alpha-1}a_2|h'(a_2)|+\frac{1}{12}a_1a_2^{\alpha-1}|h'(a_2)|+\frac{1}{4}a_2^{\alpha}|h'(a_2)|\Bigg]$$
$$=(a_2-a_1)\Bigg[\frac{a_1^{\alpha}|h'(a_1)|+a_2^{\alpha}|h'(a_2)|}{4}+a_1^{\alpha-1}a_2\left\{\frac{|h'(a_1)|+|h'(a_2)|}{12}\right\}$$
$$+a_1a_2^{\alpha-1}\left\{\frac{|h'(a_1)|+|h'(a_2)|}{12}\right\}+\frac{a_1^{\alpha}|h'(a_2)|+a_2^{\alpha}|h'(a_1)|}{12}\Bigg]. \tag{15}$$

By using (15) in (13), we obtain (9).

Now, we prove the inequality (10), by similar a procedure:

$$\int_{a_1}^{a_2} a_1^{\alpha}(a_2-s)|h'(s)|ds = (a_2-a_1)^2a_1^{\alpha}\int_0^1(1-\lambda)|h'(a_2\lambda+(1-\lambda)a_1)|d\lambda$$
$$\le (a_2-a_1)^2a_1^{\alpha}\int_0^1(1-\lambda)[\lambda|h'(a_2)|+(1-\lambda)|h'(a_1)|]d\lambda$$
$$=(a_2-a_1)^2a_1^{\alpha}\left(\frac{|h'(a_2)|+2|h'(a_1)|}{6}\right). \tag{16}$$

By using (16) in (13), we obtain (10).

Now, we prove the inequality (11). Since

$$
\begin{aligned}
&\frac{1}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}} (s^\alpha - a_1^\alpha)|h'(s)|ds \\
&= \frac{b_2-a_1}{a_2^\alpha - a_1^\alpha}\int_{\frac{1}{2}}^{1} ((a_1\lambda+(1-\lambda)a_2)^{\alpha-1}(a_1\lambda+(1-\lambda)a_2)-a_1^\alpha)|h'(a_1\lambda+(1-\lambda)a_2)|d\lambda \\
&\le \frac{a_2-a_1}{a_2^\alpha - a_1^\alpha}\int_{\frac{1}{2}}^{1} ((a_1^{\alpha-1}\lambda+(1-\lambda)a_2^{\alpha-1})(a_1\lambda+(1-\lambda)a_2)-a_1^\alpha)[\lambda|h'(a_1)|+(1-\lambda)|h'(a_2)|]d\lambda \\
&= \frac{a_2-a_1}{a_2^\alpha - a_1^\alpha}\left[\frac{15a_1^\alpha|h'(a_1)|+a_2^\alpha|h'(a_2)|}{64} + a_1^{\alpha-1}a_2\left\{\frac{11|h'(a_1)|+5|h'(a_2)|}{192}\right\}\right. \\
&\quad + a_1a_2^{\alpha-1}\left\{\frac{11|h'(a_1)|+5|h'(a_2)|}{192}\right\} + \frac{11a_1^\alpha|h'(a_2)|+5a_2^\alpha|h'(a_1)|}{192} \\
&\quad \left. - \frac{3a_1^\alpha|h'(a_2)|+a_1^\alpha|h'(a_1)|}{8}\right].
\end{aligned} \tag{17}
$$

By using (17) in (13), we obtain (11).

Now, we prove the inequality (12). Since

$$
\begin{aligned}
\frac{1}{a_2^\alpha - a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2} (a_2^\alpha - s^\alpha)|h'(s)|ds &= \frac{a_2-a_1}{a_2^\alpha - a_1^\alpha}\int_{\frac{1}{2}}^{1} (a_2^\alpha - (\lambda a_2+(1-\lambda)a_1)^\alpha)|h'(\lambda a_2+(1-\lambda)a_1)|d\lambda \\
&\le \frac{a_2-a_1}{a_2^\alpha - a_1^\alpha}\int_{\frac{1}{2}}^{1} (a_2^\alpha - (\lambda a_2^\alpha+(1-\lambda)a_1^\alpha))|h'(\lambda a_2+(1-\lambda)a_1)|d\lambda \\
&\le (a_2-a_1)\int_{\frac{1}{2}}^{1} (\lambda(1-\lambda)|h'(a_2)|+(1-\lambda)^2|h'(a_1)|)d\lambda \\
&= (a_2-a_1)\left[\frac{2|h'(a_2)|+|h'(a_1)|}{24}\right].
\end{aligned} \tag{18}
$$

By using (18) in (13), we obtain (12). □

In the last theorem we obtain new bounds for the left difference of Hermite-Hadamard inequality by using concavity of $|h'|$.

Theorem 3. *Let $a_1, a_2 \in \mathbf{R}^+$ and $h : [a_1, a_2] \to \mathbf{R}$ be a differentiable function such that $|h'|$ is concave. Then for $\alpha \in (0,1]$, the following inequalities hold:*

$$
\begin{aligned}
|\Delta(h;a_1,a_2)| &\le \frac{a_2^\alpha(a_2-a_1)}{2(a_2^\alpha - a_1^\alpha)}\left|h'\left(\frac{a_1+2a_2}{3}\right)\right| + \frac{1}{(a_2^\alpha - a_1^\alpha)(a_2-a_1)}\left[\int_{a_1}^{a_2} s^\alpha(a_2-a_1)|h'(s)|ds\right. \\
&\quad \left. + \int_{a_1}^{a_2} a_1^\alpha(a_2-s)|h'(s)|ds\right] + \frac{1}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}} (s^\alpha - a_1^\alpha)|h'(s)|ds \\
&\quad + \frac{1}{a_2^\alpha - a_1^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2} (a_2^\alpha - s^\alpha)|h'(s)|ds.
\end{aligned} \tag{19}
$$

$$
|\Delta(h;a_1,a_2)| \le \frac{(a_2-a_1)}{(a_2^\alpha - a_1^\alpha)}\left(\frac{2a_1^\alpha + a_1^{\alpha-1}a_2 + a_1a_2^{\alpha-1} + 2a_2^\alpha}{6}\right)
$$

$$\times \left|h'\left(\frac{3a_1^{\alpha+1}+2a_1^{\alpha}a_2+2a_1a_2^{\alpha-1}+a_1^2a_2^{\alpha-1}+a_1^{\alpha-1}a_2^2+3a_2^{\alpha+1}}{2(2a_1^{\alpha}+a_1^{\alpha-1}a_2+a_1a_2^{\alpha-1}+2a_2^{\alpha})}\right)\right|$$
$$+\frac{1}{(a_2^{\alpha}-a_1^{\alpha})(a_2-a_1)}\left[\int_{a_1}^{a_2}a_2^{\alpha}(s-a_1)|h'(s)|ds+\int_{a_1}^{a_2}a_1^{\alpha}(a_2-s)|h'(s)|ds\right]$$
$$+\frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^{\alpha}-a_1^{\alpha})|h'(s)|ds+\frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2^{\alpha}-s^{\alpha})|h'(s)|ds. \tag{20}$$

Proof. From Lemma 2 and using the property of the modulus, we can write

$$|\Delta(h;a_1,a_2)|$$
$$\le\frac{1}{(a_2^{\alpha}-a_1^{\alpha})(a_2-a_1)}\left[\int_{a_1}^{a_2}a_2^{\alpha}(s-a_1)|h'(s)|ds+\int_{a_1}^{a_2}s^{\alpha}(a_2-a_1)|h'(s)|ds+\int_{a_1}^{a_2}a_1^{\alpha}(a_2-s)|h'(s)|ds\right]$$
$$+\frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{\frac{a_1+a_2}{2}}(s^{\alpha}-a_1^{\alpha})|h'(s)|ds+\frac{1}{a_2^{\alpha}-a_1^{\alpha}}\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2^{\alpha}-s^{\alpha})|h'(s)|ds. \tag{21}$$

Now, by change of variable and using Jensen's integral inequality we obtain

$$\int_{a_1}^{a_2}(s-a_1)|h'(s)|ds\le(a_2-a_1)^2\left(\int_0^1(1-\lambda)d\lambda\right)\left|h'\left(\frac{\int_0^1(1-\lambda)(\lambda a_1+(1-\lambda)a_2)d\lambda}{\int_0^1(1-\lambda)d\lambda}\right)\right|$$
$$=\frac{(a_2-a_1)^2}{2}\left|h'\left(\frac{a_1+2a_2}{3}\right)\right|. \tag{22}$$

By using (22) in (21), we obtain (19).

Now, we prove the inequality (20).

Similarly, as above by change of variable and using Jensen's integral inequality, we have

$$\int_{a_1}^{a_2}s^{\alpha}|h'(s)|ds$$
$$=(a_2-a_1)\int_0^1(a_1\lambda+(1-\lambda)a_2)^{\alpha}|h'(a_1\lambda+(1-\lambda)a_2)|d\lambda$$
$$=(a_2-a_1)\int_0^1(a_1\lambda+(1-\lambda)a_2)^{\alpha-1}(a_1\lambda+(1-\lambda)a_2)|h'(a_1\lambda+(1-\lambda)a_2)|d\lambda$$
$$\le(a_2-a_1)\int_0^1(a_1^{\alpha-1}\lambda+(1-\lambda)a_2^{\alpha-1})(a_1\lambda+(1-\lambda)a_2)|h'(a_1\lambda+(1-\lambda)a_2)|d\lambda$$
$$=(a_2-a_1)\int_0^1(a_1^{\alpha}\lambda^2+a_1^{\alpha-1}\lambda(1-\lambda)a_2+\lambda(1-\lambda)a_1a_2^{\alpha-1}+(1-\lambda)^2a_2^{\alpha})|h'(a_1\lambda+(1-\lambda)a_2)|d\lambda$$
$$\le(a_2-a_1)\left(\int_0^1(a_1^{\alpha}\lambda^2+a_1^{\alpha-1}\lambda(1-\lambda)a_2+\lambda(1-\lambda)a_1a_2^{\alpha-1}+(1-\lambda)^2a_2^{\alpha})d\lambda\right)$$
$$\times\left|h'\left(\frac{\int_0^1(a_1^{\alpha}\lambda^2+a_1^{\alpha-1}\lambda(1-\lambda)a_2+\lambda(1-\lambda)a_1a_2^{\alpha-1}+(1-\lambda)^2a_2^{\alpha})(a_1\lambda+(1-\lambda)a_2)d\lambda}{\int_0^1(a_1^{\alpha}\lambda^2+a_1^{\alpha-1}\lambda(1-\lambda)a_2+\lambda(1-\lambda)a_1a_2^{\alpha-1}+(1-\lambda)^2a_2^{\alpha})d\lambda}\right)\right|$$
$$=(a_2-a_1)\left(\frac{2a_1^{\alpha}+a_1^{\alpha-1}a_2+a_1a_2^{\alpha-1}+2a_2^{\alpha}}{6}\right)$$
$$\times\left|h'\left(\frac{3a_1^{\alpha+1}+2a_1^{\alpha}a_2+2a_1a_2^{\alpha-1}+a_1^2a_2^{\alpha-1}+a_1^{\alpha-1}a_2^2+3a_2^{\alpha+1}}{2(2a_1^{\alpha}+a_1^{\alpha-1}a_2+a_1a_2^{\alpha-1}+2a_2^{\alpha})}\right)\right|. \tag{23}$$

By using (23) in (21), we obtain (20). □

3. Application to Means

There are many application of means in the real life. For example, if R_1 and R_2 are the resistances of two series combination of resistor, then the total resistance is computed by the formula:

$$R_T = R_1 + R_2 = 2A(R_1, R_2),$$

which is double the arithmetic mean. Similarly, applications of the harmonic mean in Asian options of stock can be found in [1]. We will consider the following particular means for any $a_1, a_2 \in \mathbf{R}$, $a_1 \neq a_2$, which are well-known in the literature, see [13]:

$$A(a_1,a_2) = \frac{a_1+a_2}{2}, \quad a_1,a_2>0, \text{ the arithmetic mean},$$

$$L(a_1,a_2) = \frac{a_2-a_1}{\ln a_2 - \ln a_1}, \quad a_1,a_2>0, \text{ the logarithmic mean},$$

$$L_\mu(a_1,a_2) = \left[\frac{a_2^{\mu+1}-a_1^{\mu+1}}{(\mu+1)(a_2-a_1)}\right]^{\frac{1}{\mu}}, \quad a_1<a_2, \mu \in \mathbf{R},$$

the Stolarsky mean and $\lim_{\mu\to -1} L_\mu(a_1,a_2) = L(a_1,a_2)$.

In the following proposition we give applications of Theorem 1 for means.

Proposition 1. *Let $0<a_1<a_2$ and $\mu<0$. Then we have the following inequality*

$$\begin{aligned}
&L_\mu(a_1,a_2)^\mu - A^\mu(a_1,a_2)\\
&\geq \frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\left(\frac{\mu(a_2-a_1)(a_2^{\alpha+\mu}-a_1^{\alpha+\mu})}{\alpha+\mu} - \mu a_1^\alpha\left\{\frac{a_2(a_2^\mu-a_1^\mu)}{\mu} - \frac{a_2^{\mu+1}-a_1^{\mu+1}}{\mu+1}\right\}\right)\\
&\quad - \frac{\mu}{a_2^\alpha-a_1^\alpha}\left(\frac{1}{\alpha+\mu}\left\{\left(\frac{a_1+a_2}{2}\right)^{\mu+\alpha} - a_1^{\alpha+\mu}\right\} - \frac{a_1^\alpha}{\mu}\left\{\left(\frac{a_1+a_2}{2}\right)^\mu - a_1^\mu\right\}\right)\\
&\quad + \frac{\mu}{a_2^\alpha-a_1^\alpha}\left(\frac{a_2^\alpha}{\mu}\left\{a_2^\mu - \left(\frac{a_1+a_2}{2}\right)^\mu\right\} - \frac{1}{\alpha+\mu}\left\{a_2^{\alpha+\mu} - \left(\frac{a_1+a_2}{2}\right)^{\alpha+\mu}\right\}\right).
\end{aligned}$$

Proof. We choose $h_1(s) = s^\mu$, $s>0$, $\mu<0$. Since $h_1'(s) = \mu s^{\mu-1} < 0$, $h_1(s)$ is decreasing. Now, by using h_1 in (6), we deduce the required result. □

In the following proposition we give applications of Theorem 2 for means.

Proposition 2. *Let $0<a_1<a_2$ and $\mu>2$. Then we have the following inequality*

$$\begin{aligned}
&|A^\mu(a_1,a_2) - L_\mu(a_1,a_2)^\mu|\\
&\leq \frac{\mu a_2^\alpha(a_2-a_1)}{(a_2^\alpha-a_1^\alpha)}\left\{\frac{|a_1|^{\mu-1}+2|a_2|^{\mu-1}}{6}\right\}\\
&\quad + \frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\left(\frac{\mu(a_2-a_1)(a_2^{\alpha+\mu}-a_1^{\alpha+\mu})}{\alpha+\mu} + \mu a_1^\alpha\left\{\frac{a_2(a_2^\mu-a_1^\mu)}{\mu} - \frac{a_2^{\mu+1}-a_1^{\mu+1}}{\mu+1}\right\}\right)
\end{aligned}$$

$$+\frac{\mu}{a_2^\alpha-a_1^\alpha}\left(\frac{1}{\alpha+\mu}\left\{\left(\frac{a_1+a_2}{2}\right)^{\mu+\alpha}-a_1^{\alpha+\mu}\right\}-\frac{a_1^\alpha}{\mu}\left\{\left(\frac{a_1+a_2}{2}\right)^{\mu}-a_1^{\mu}\right\}\right)$$
$$+\frac{\mu}{a_2^\alpha-a_1^\alpha}\left(\frac{a_2^\alpha}{\mu}\left\{a_2^\mu-\left(\frac{a_1+a_2}{2}\right)^{\mu}\right\}-\frac{1}{\alpha+\mu}\left\{a_2^{\alpha+\mu}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+\mu}\right\}\right).$$

Proof. Consider $h_2(s)=s^\mu, s>0, \mu>2$, then $f(s):=|h_2'(s)|=\mu s^{\mu-1}$. Now $f''(s)=\mu(\mu-1)(\mu-2)s^{\mu-3}>0$. So $|h_2'|$ is convex. By using h_2 in (8), we deduce the required result. □

In the following proposition we give applications of Theorem 3 for means.

Proposition 3. *Let $0<a_1<a_2$ and $1<\mu<2$. Then we have the following inequality*

$$\begin{aligned}
&|A^\mu(a_1,a_2)-L_\mu(a_1,a_2)^\mu|\\
&\le\frac{\mu a_2^\alpha(a_2-a_1)}{2(a_2^\alpha-a_1^\alpha)}\left|\frac{a_1+2a_2}{3}\right|^{\mu-1}\\
&\quad+\frac{1}{(a_2^\alpha-a_1^\alpha)(a_2-a_1)}\left(\frac{\mu(a_2-a_1)(a_2^{\alpha+\mu}-a_1^{\alpha+\mu})}{\alpha+\mu}+\mu a_1^\alpha\left\{\frac{a_2(a_2^\mu-a_1^\mu)}{\mu}-\frac{a_2^{\mu+1}-a_1^{\mu+1}}{\mu+1}\right\}\right)\\
&\quad+\frac{\mu}{a_2^\alpha-a_1^\alpha}\left(\frac{1}{\alpha+\mu}\left\{\left(\frac{a_1+a_2}{2}\right)^{\mu+\alpha}-a_1^{\alpha+\mu}\right\}-\frac{a_1^\alpha}{\mu}\left\{\left(\frac{a_1+a_2}{2}\right)^{\mu}-a_1^{\mu}\right\}\right)\\
&\quad+\frac{\mu}{a_2^\alpha-a_1^\alpha}\left(\frac{a_2^\alpha}{\mu}\left\{a_2^\mu-\left(\frac{a_1+a_2}{2}\right)^{\mu}\right\}-\frac{1}{\alpha+\mu}\left\{a_2^{\alpha+\mu}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+\mu}\right\}\right).
\end{aligned}$$

Proof. Consider $h_3(s)=s^\mu$, $s>0$, $1<\mu<2$, then $f(s):=|h_3'(s)|=\mu s^{\mu-1}$. Now, $f''(s)=\mu(\mu-1)(\mu-2)s^{\mu-3}<0$. So $|h_3'|$ is concave. By using h_3 in (19), we deduce the required result. □

Remark. The remaining parts of Theorems 1-3, can analogously be applied for h_1,h_2,h_3 respectively to obtain the related results of Propositions 1-3.

4. Conclusion

In this paper we have used Montgomery identity and obtained an integral identity for the difference of Hermite-Hadamard inequality. By using this identity for different classes of monotone and convex functions we have established several Hermite-Hadamard type inequalities. We have also applied the results for particular functions and deduced inequalities for arithmetic and generalized logarithmic means.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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