



On Some Properties of Generalized Fock Space $F^2(dv_\alpha)$ by Frame Theory on the C^n

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Abstract. We obtain a short and new sharp proof for atomic decomposition for Fock space on the C^n by using the frame theory. In fact we show that the generalized Fock space $F^2(dv_\alpha)$ on the C^n admit an atomic decomposition i.e every analytic function in this space can be presented as a linear combination of “atoms” defined using the normalized reproducing kernel of this space.

1. Introduction

The decomposition of an element of a Banach space on a domain is a widely studied area of modern mathematics of which atomic decomposition is an example. An atomic decomposition consists of a sequence of simple building blocks (called atoms), such that every element is a linear combination $\sum_m a_m k(\lambda_m)$ of atoms k with $\sum_m |a_m|^p < \infty$ for some $1 \leq p \leq \infty$, $a_m \in C$. The infimum of the sum of the coefficients a_m defines the norm or an equivalent one for the Banach space. Thus an atomic decomposition is a sequence which has basis-like properties but which does not need to be a basis.

In general atomic decompositions are overcomplete, the sampling sequences $\{\lambda_m\}$ usually contain too many points for the set of atoms $\{k(\lambda_m)\}$ to be linearly independent in which case it forms a frame instead of a basis.

First to come up with the idea of atomic decomposition were Coifman and Rochberg [4] who in 1980 showed that a “decomposition theorem” holds for domains in the Bergman space $A^p(D, dv)$ of analytic functions on a bounded symmetric domain $D \subset C^n$.

Throughout this paper, let $z = (z_1, z_2, \dots, z_n)$ and $w = (w_1, w_2, \dots, w_n)$ be points in C^n , we write $\langle z, w \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2 + \dots + z_n \bar{w}_n$ and let dv denote the usual Lebesgue volume measure on C^n and for $\alpha > 0$, dv_α is defined as follow

$$dv_\alpha(z) = \frac{\alpha^n}{\pi} e^{-\alpha|z|^2} dv(z),$$

then the Fock space $F^2(dv_\alpha)$ is the space of all entire functions f on C^n for which

$$\|f\|_{F^2(dv_\alpha)}^2 = \int_{C^n} |f(z)|^2 dv_\alpha(z) < \infty.$$

$F^2(dv_\alpha)$ is a Hilbert space with an inner product

$$\langle f, g \rangle_{F^2(dv_\alpha)} = \int_{C^n} f(z) \overline{g(z)} dv_\alpha(z).$$

Since each point evaluation is a bounded linear functional on $F^2(dv_\alpha)$, then the Riesz representation theorem implies that there exists a unique function $K_{n,z}^\alpha$ in $F^2(dv_\alpha)$ such that

$$f(z) = \int_{C^n} f(w) \overline{K_{n,z}^\alpha(w)} dv_\alpha(w), \quad \text{for all } f \in F^2(dv_\alpha). \quad (1.1)$$

Let $K_n^\alpha(z, w)$ be the function on $C^n \times C^n$ defined by

$$K_n^\alpha(z, w) = \overline{K_{n,z}^\alpha(w)}.$$

$K_n^\alpha(z, w)$ is called the Fock kernel or the reproducing kernel of $F^2(dv_\alpha)$.

The formula for $K_{n,z}^\alpha$ by [5] is:

$$K_{n,z}^\alpha(w) = \exp\{\alpha \langle w, z \rangle\}, \quad w \in C^n.$$

Here the notation $\langle \cdot, \cdot \rangle$ denotes the usual inner product on C^n . For any $w \in C^n$.

Let

$$k_{n,z}^\alpha(w) = \frac{K_n^\alpha(w, z)}{\sqrt{K_n^\alpha(z, z)}} = \exp\left\{\alpha \langle w, z \rangle - \frac{\alpha}{2} |z|^2\right\}.$$

$k_{n,z}^\alpha$ are called the normalized reproducing Kernel of $F^2(dv_\alpha)$.

By [7] a sequence $\{\lambda_m\}$ of a complex numbers is a sampling sequence for $F^2(dv_\alpha)$ if there exist positive constant A and B such that

$$A \|f\|_{F^2(dv_\alpha)}^2 \leq \sum_{m=1}^{\infty} e^{-\alpha |\lambda_m|^2} |f(\lambda_m)|^2 \leq B \|f\|_{F^2(dv_\alpha)}^2.$$

Many operator-theoretic problems in the analysis of Fock space involve estimating integral operators whose kernel is a power of the Fock kernel. This together with the use of the reproducing property of the Fock kernel brings us to a close relative of the formula (1.1), the atomic decomposition.

The integral in (1.1) is then approximated by a Riemannian sum over the partition using the values of f and the kernel K at the points λ_m of the lattice. If partition is suitable, this will produce a good approximation.

In search for a representation of f as a linear combination of atoms this makes sense, since the kernels $k_{n,z}^\alpha$ are also the unit vectors in $F^2(dv_\alpha)$ and, in some sense, play the part of an orthonormal basis for $F^2(dv_\alpha)$ even though they are not mutually orthogonal.

The atomic decomposition of the function $f \in F^2(dv_\alpha)$:

$$f(z) = \sum_{m=1}^{\infty} a_m \frac{e^{\alpha \langle z, \lambda_m \rangle}}{\sqrt{e^{\alpha |\lambda_m|^2}}} \quad (1.2)$$

valid such that for any $\{a_m\} \in \ell^2$, the function in (1.2) is in $F^2_\alpha(dv_\alpha)$ and if $f \in F^2(dv_\alpha)$, then there is a sequence $\{a_m\} \in \ell^2$ such that (1.2) holds. For the coefficients we get

$$\sum_{m=1}^{\infty} |a_m|^2 < \infty \quad \text{and} \quad \|\{a_m\}\|_{\ell^2} \asymp \|f\|_{F^2(dv_\alpha)}. \tag{1.3}$$

In Fock space the atomic decomposition can thus be regarded as a discrete analogue of the reproducing property, where it was derived from initially. The utility of an atomic decomposition is that it is often possible to prove statements about $F^2(dv_\alpha)$ by verifying them first in the simple special case of atoms and then extending the results to the entire space. An immediate corollary of the atomic decomposition is that it establishes an isomorphism between $F^2(dv_\alpha)$ and the sequence space ℓ^2 .

2. Frames Theory

The Frame theory was introduced by Duffin and Schaeffer [6] in order to establish general conditions under which one can reconstruct perfectly a function f in a Hilbert space H from its inner product $(\langle \cdot, \cdot \rangle_H)$ with a family of vectors $\{f_n\}_{n \in I}$ where I can be a finite or infinite countable index set.

2.1. Definition. A set of vectors $\{f_n\}_{n \in I}$ is a frame of Hilbert space H if there exists two constants $A, B > 0$ so that

$$A\|f\|_H^2 \leq \sum_{n \in I} |\langle f, f_n \rangle_H|^2 \leq B\|f\|_H^2, \quad \text{for all } f \in H$$

The number A, B are called frame bounds. They are not unique.

2.2. Definition. A set of vectors $\{f_n\}_{n \in I}$ is called Bessel sequence if there exist a constant $B > 0$ such that

$$\sum_{n \in I} |\langle f, f_n \rangle_H|^2 \leq B\|f\|_H^2, \quad \text{for all } f \in H$$

2.3. Theorem. Let $\{f_n\}_{n \in I} \subset H$. Then $\{f_n\}_{n \in I}$ is a Bessel sequence with Bessel bound B if and only if the mapping $T : \{a_n\} \rightarrow \sum_{n=1}^{\infty} a_n f_n$ is a well defined operator and bounded from ℓ^2 onto H and $\|T\| \leq \sqrt{B}$.

Proof. See [3]. □

Since a frame $\{f_n\}_{n \in I}$ is a Bessel sequence, the operator $T : \ell^2 \rightarrow H$ by $T(\{a_n\}) = \sum_{n=1}^{\infty} a_n f_n$ is bounded and linear. T is sometimes called the preframe operator.

2.4. Theorem. Let $\{f_n\}_{n \in I} \subset H$. The following two statements are equivalent:

- (1) $\{f_n\}$ is a frame with bounds A, B .
- (2) $Sf = \sum_{n \in I} \langle f, f_n \rangle f_n$ is a bounded and invertible linear operator on H , with $AI \leq S \leq BI$.

Proof. See [3]. □

2.5. Theorem. A sequence $\{f_n\}_{n \in I} \subset H$ is a frame for H if and only if the mapping

$$T : \{a_n\} \rightarrow \sum_{n=1}^{\infty} a_n f_n$$

is a well defined mapping of ℓ^2 onto H .

Proof. See [2]. □

2.6. Definition. Let the sequence $\{f_n\}_{n \in I}$ be a frame with bounds A, B , then operator $S : H \rightarrow H$ by $Sf = \sum_{n \in I} \langle f, f_n \rangle f_n$ is called the frame operator. since $AI \leq S \leq BI$ we have

$$\|I - B^{-1}S\| \leq \left\| \frac{B-A}{B} \right\| < 1,$$

which by the Neumann theorem shows that S is invertible. If S^{-1} commutes with both S and I , and multiplying $AI \leq S \leq BI$ with S^{-1} yields the bellow inequality

$$B^{-1}I \leq S^{-1} \leq A^{-1}I,$$

thus S^{-1} is bounded. Also S is onto (see [1]).

3. Atomic decomposition

The purpose of this section is to present a short and new sharp proof for atomic decomposition for Fock spaces and show normalized reproducing kernels are building blocks for $F^2(dv_\alpha)$. In some sense, they play the role of an orthonormal basis for this space, although they are clearly not mutually orthogonal. At first we prove following lemma.

3.1. Lemma. $\{\lambda_m\}$ is a sampling sequence for $F^2(dv_\alpha)$ if and only if the set of normalized reproducing kernels $\{k_{n, \lambda_m}^\alpha\}$ is a frame for $F^2(dv_\alpha)$.

Proof. For $f \in F^2_\alpha(dv_\alpha)$

$$\begin{aligned} \sum_{m=1}^{\infty} |\langle f, k_{n, \lambda_m}^\alpha \rangle|^2 &= \sum_{m=1}^{\infty} \left| \int_{C^n} f(z) \overline{k_{n, \lambda_m}^\alpha(z)} dv_\alpha(z) \right|^2 \\ &= \sum_{m=1}^{\infty} \left| \int_{C^n} f(z) \frac{e^{\alpha \langle z, \lambda_m \rangle}}{\sqrt{e^{\alpha |\lambda_m|^2}}} dv_\alpha(z) \right|^2 \\ &= \sum_{m=1}^{\infty} e^{-\alpha |\lambda_m|^2} \left| \int_{C^n} f(z) e^{\alpha \langle z, \lambda_m \rangle} dv_\alpha(z) \right|^2 \\ &= \sum_{m=1}^{\infty} e^{-\alpha |\lambda_m|^2} |f(\lambda_m)|^2. \end{aligned} \quad \square$$

Let $\{\lambda_m\}$ be a sampling sequence for $F^2(dv_\alpha)$, we define two operators

$$\begin{aligned} T_n^\alpha &: \ell^2 \rightarrow F^2(dv_\alpha) \\ T_n^\alpha(\{a_m\})(z) &= \sum_{m=1}^{\infty} a_m \frac{e^{\alpha\overline{(z, \lambda_m)}}}{\sqrt{e^{\alpha|\lambda_m|^2}}} \\ S_n^\alpha &: F^2(dv_\alpha) \rightarrow F^2(dv_\alpha) \\ S_n^\alpha(f(z)) &= \sum_{m=1}^{\infty} e^{-\alpha|\lambda_m|^2} f(\lambda_m) e^{\alpha\overline{\lambda_m}}. \end{aligned}$$

3.2. Lemma. T^α is bounded and onto.

Proof. Since $\{k_{n, \lambda_m}^\alpha\}$ is a frame for $F^2(dv_\alpha)$, so

$$T^\alpha(\{a_m\})(z) = \sum_{m=1}^{\infty} a_m k_{n, \lambda_m}^\alpha(z) = \sum_{m=1}^{\infty} a_m \frac{e^{\alpha\overline{(z, \lambda_m)}}}{\sqrt{e^{\alpha|\lambda_m|^2}}}.$$

Thus by theorem (2.3) T_n^α is bounded and onto □

3.3. Theorem. S_n^α is bounded and invertible.

Proof. Since $\{k_{n, \lambda_m}^\alpha\}$ is a frame for $F^2(dv_\alpha)$, so

$$\begin{aligned} S_n^\alpha(f(z)) &= \sum_{m=1}^{\infty} \langle f, k_{n, \lambda_m}^\alpha \rangle k_{n, \lambda_m}^\alpha(z) \\ &= \sum_{m=1}^{\infty} \int_{C^n} f(z) \overline{k_{n, \lambda_m}^\alpha(z)} dv_\alpha(z) k_{n, \lambda_m}^\alpha(z) \\ &= \sum_{m=1}^{\infty} \int_{C^n} f(z) \frac{e^{\alpha\overline{(z, \lambda_m)}}}{e^{\sqrt{\alpha}|\lambda_m|}} dv_\alpha(z) k_{n, \lambda_m}^\alpha(z) \\ &= \sum_{m=1}^{\infty} e^{-\frac{\alpha|\lambda_m|^2}{2}} f(\lambda_m) \frac{e^{\alpha\overline{(z, \lambda_m)}}}{\sqrt{e^{\alpha|\lambda_m|^2}}} \\ &= \sum_{m=1}^{\infty} e^{-\alpha|\lambda_m|^2} f(\lambda_m) e^{\alpha\overline{(z, \lambda_m)}}. \end{aligned}$$

Thus by theorem (2.4) S_n^α is bounded and invertible. □

Finally we state the atomic decomposition theorem for the Fock space.

3.4. Theorem. There exists a sequence $\{\lambda_m\}$ in C^n and constant $c > 0$ with the following properties:

(1) For any $\{a_m\}$ in ℓ^2 , the function

$$f(z) = \sum_{m=1}^{\infty} a_m \frac{e^{\alpha\overline{(z, \lambda_m)}}}{\sqrt{e^{\alpha|\lambda_m|^2}}}$$

is in $F^2(dv_\alpha)$ with

$$\|f\|_{F^2(dv_\alpha)} \leq c \|\{a_n\}\|_{\ell^2}$$

(2) For any $f \in F^2(dv_\alpha)$, there is $\{a_m\} \in \ell^2$ such that,

$$f(z) = \sum_{m=1}^{\infty} a_m \frac{e^{\alpha\langle z, \lambda_m \rangle}}{\sqrt{e^{\alpha|\lambda_m|^2}}}$$

and $\|\{a_m\}\|_{\ell^2} \leq c \|f\|_{F^2(dv_\alpha)}$.

Proof. Let $\{\lambda_m\}$ be a sampling sequence for $F^2(dv_\alpha)$

(1) By definition of T_n^α , it is clear that:

$$f(z) = \sum_{m=1}^{\infty} a_m \frac{e^{\alpha\langle z, \lambda_m \rangle}}{\sqrt{e^{\alpha|\lambda_m|^2}}}$$

is in $F^2(dv_\alpha)$. Also by theorem (2.3), T_n^α is bounded. Thus

$$\|f\|_{F^2(dv_\alpha)} \leq c \|\{a_m\}\|_{\ell^2}.$$

(2) By definition (2.6) S_n^α is onto. Thus if $f \in F^2(dv_\alpha)$ then there exist $g \in F^2(dv_\alpha)$ such that

$$\begin{aligned} f(z) &= S_n^\alpha g(z) \\ &= \sum_{m=1}^{\infty} e^{-\alpha|\lambda_m|^2} g(\lambda_m) e^{\alpha\langle z, \lambda_m \rangle} \\ &= \sum_{m=1}^{\infty} e^{\frac{-\alpha|\lambda_m|^2}{2}} g(\lambda_m) \frac{e^{\alpha\langle z, \lambda_m \rangle}}{\sqrt{e^{\alpha|\lambda_m|^2}}} \\ &= \sum_{m=1}^{\infty} a_m \frac{e^{\alpha\langle z, \lambda_m \rangle}}{\sqrt{e^{\alpha|\lambda_m|^2}}} \end{aligned}$$

such that a_m defined by $a_m = e^{\frac{-\alpha|\lambda_m|^2}{2}} g(\lambda_m)$.

Now the second part of (2) by the boundedness of $(S_n^\alpha)^{-1}$ and definition of sampling sequence is proved by

$$\begin{aligned} \|\{a_m\}\|_{\ell^2} &= \left(\sum_{m=1}^{\infty} |a_m|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{m=1}^{\infty} e^{-\alpha|\lambda_m|^2} |g(\lambda_m)|^2 \right)^{\frac{1}{2}} \\ &\leq (c_1 \|g\|_{F^2(dv_\alpha)}^2)^{\frac{1}{2}} \\ &= c_1 \|g\|_{F^2(dv_\alpha)} \\ &= c_1 \|(S_n^\alpha)^{-1} f\|_{F^2(dv_\alpha)} \\ &\leq c \|f\|_{F^2(dv_\alpha)}. \quad \square \end{aligned}$$

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Received May 11, 2010

Accepted May 27, 2010