Communications in Mathematics and Applications
Vol. 10, No. 1, pp. 71–84, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i1.1148

Research Article

Roughness Applied to Generalized Γ-ideals of Ordered LA Γ-ideals

Moin Akhtar Ansari

Department of Mathematics, College of Science, Main Campus, Post Box 2097,
Jazan University, Jazan, Kingdom of Saudi Arabia
maansari@jazanu.edu.sa

Abstract. In this paper, we have applied rough set theory to ordered Γ-ideals mainly quasi(bi)-Γ-ideals in ordered LA Γ-semigroups by using pseudoorder of relations. Based on this relation we have studied properties of rough ideals, rough interior ideals, rough quasi-ideals, rough bi-ideals and rough prime ideals in ordered LA-Γ-semigroups and have shown that lower and upper approximations of an LA-Γ-subsemigroups (resp., left ideals, right ideals, interior ideals, two-sided ideals, quasi-ideals, bi-ideals and prime ideals) in an ordered LA-Γ-semigroup is an LA-Γ subsemigroups (resp., left ideals, right ideals, interior ideals, two-sided ideals, quasi-ideals, bi-ideals and prime ideals).

Keywords. Ordered LA-Γ-semigroups; Rough sets; Rough (left, right, quasi, bi, prime and interior)-Γ-ideals

MSC. 20M10; 20M12; 20N99

Received: December 10, 2018 **Accepted:** February 8, 2019

1. Introduction

The notion of rough sets was introduced by Pawlak in his paper [19]. The rough set theory has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy or incomplete information. In connection with algebraic structures, Biswas and Nanda [6] introduced the notion of rough subgroups, whereas Kuroki [16] introduced it for semigroups. Rough prime (m,n) bi-ideals in semigroups was investigated by Yaqoob *et al*. [26] and studied in case of rough fuzzy prime bi-ideals in semigroups [27]. Aslam *et al*.

[5] presented some results on roughness in semigroups. Xiao and Zhang [25] studied rough prime ideals and rough fuzzy prime ideals in semigroups. In connection to logical algebras, mainly in KU-algebras Moin and Ali [2] have introduced and studied rough approximations in KU-algebras whereas rough set theory has been applied to UP-algebras by Moin *et al*. [4] and their algebraic properties have been studied therein.

The concept of an AG-groupoid was first given by Kazim and Naseeruddin [10] in 1972 and they called it left almost semigroups (LA-semigroups). Holgate [9] called LA-semigroup to left invertive groupoid. The concept of a Γ-semigroup (generalization of semigroups) was introduced by M.K. Sen [22] in 1981 as follows: A nonempty set S is called a Γ-semigroup if the following assertions are satisfied: (1) $a\alpha b \in S$ and $\alpha a\beta \in \Gamma$ and (2) $(a\alpha b)\beta c = a\alpha(b\beta)c = a\alpha(b\beta c)$ for all $a,b,c \in S$ and all $\alpha,\beta \in \Gamma$. But further in 1986, M.K. Sen and N.K. Saha [21] defined Γ-semigroups in different way as: Given two nonempty sets M and Γ, M is called a Γ-semigroup if (1) $a\alpha b \in M$ and (2) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c \in M$ and all $\alpha,\beta \in \Gamma$. They defined Γ-groupoid in the same paper [21]. Further ordered Γ-semigroup was explained by Dutta *et al*. [7] and then investigated by Kehayopulu [11] whereas ordered Γ-abel-grassmann's groupoids was undertaken by [14]. In some direction of fuzziness ordered AG-groupoids has been studied by Faisal *et al*. [8]. Ordered LA-semigroup has been taken under consideration in terms of interval valued fuzzy ideals by Asghar Khan *et al*. [13]. It should be noted that an ordered Γ-AG-groupoid is the generalization of an ordered Γ-semigroups.

An LA-Γ-semigroup is a groupoid having the left invertive law

$$(a\alpha b)\beta c = (c\alpha b)\beta a, \text{ for all } a,b,c \in S \text{ and for all } \alpha,\beta \in \Gamma.$$

In an LA-Γ-semigroup, the medial law holds

$$(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d), \text{ for all } a,b,c,d \in S \text{ and for all } \alpha,\beta,\gamma \in \Gamma.$$

An LA-semigroup with right identity becomes a commutative monoid [18]. The connection of a commutative inverse semigroup with an LA-semigroup has been given in [17] as, a commutative inverse semigroup $(S,\circ)$ becomes an LA-semigroup $(S,\cdot)$ under $a\cdot b = b\circ a^{-1}$, for all $a,b \in S$. A commutative semigroup with identity comes from LA-semigroup by the use of a right identity. The concept of an ordered LA-semigroup was introduced by Shah *et al*. [24] and further it was extended to the theory of fuzzy sets in ordered LA-semigroups [12]. Generalized roughness in $(\in,\in \vee qk)$ have been studied by Muhammad *et al*. [1]. T-roughness and its ideals in ternary semigroups were introduced in [3].

The concept of an ordered LA-semigroup was introduced by Shah *et al*. [24] and further it was extended to the theory of fuzzy sets in ordered LA-semigroups [12]. Generalized roughness in $(\in,\in \vee qk)$ have been studied by Muhammad *et al*. [1]. Recently, generalized roughness in LA-Semigroups was studied by Noor *et al*. [20].

Let S and Γ be two nonempty sets. Then a triplet $(S,\Gamma,\cdot)$ is called an LA-Γ-semigroup, where $\cdot$ is a ternary operation $S\times\Gamma\times S \Rightarrow S$ such that $(x\cdot\alpha\cdot y)\cdot\beta\cdot z = (z\cdot\alpha\cdot y)\cdot\beta\cdot x$ for all $x,y,z \in S$ and all $\alpha,\beta \in \Gamma$ [23]. An ordered LA-Γ-semigroup is an ordered set $(S,\le)$ at the same time an LA-Γ-semigroup $(S,\Gamma,\cdot)$ such that $a \le b \Rightarrow a\cdot\alpha\cdot x \le b\cdot\alpha\cdot x$ and $x\cdot\beta\cdot a \le x\cdot\beta\cdot b$ $\forall\ a,b,x \in S$ and

$\alpha,\beta \in \Gamma$. For the sake of convenience we write $a \cdot \alpha \cdot x \le b \cdot \alpha \cdot x$ as $a\alpha x \le b\alpha x$. In what follows we denote the ordered LA-Γ-semigroup $(S,\Gamma,\cdot,\le)$ by S unless otherwise specified.

In this paper we use pseudoorder to define and investigate lower and upper approximations of an ordered LA-Γ semigroups. Rough approximations through pseudoorder of relations in algebraic structures like ordered LA-Γ-semigroups (similar other structures), make roughness more stronger than comparatively a linear order and somehow brings it more closer to an interval valued oriented rather than a linear valued orientation which actually reflect it towards fuzzyfication and many other directions of modern research.

We prove that the lower and upper approximations of an LA-Γ-subsemigroups (resp., left ideals, right ideals, interior ideals, two-sided ideals, quasi-ideals, bi-ideals and prime ideals) in an ordered LA-Γ-semigroup is an LA-Γ subsemigroups (resp., left ideals, right ideals, interior ideals, two-sided ideals, quasi-ideals, bi-ideals and prime ideals).

In Section 2, we give the basic definitions and examples that are useful for the results shown in this paper. In Section 3, we define rough approximations in ordered LA Γ-semigroups. In Section 4, we give the concepts of rough ideals (quasi-ideals, bi-ideals and interior) in ordered LA Γ-semigroups.

2. Preliminaries and Basic Definitions

Definition 2.1. An ordered LA-Γ-semigroup (po-LA-Γ-semigroup) is a structure $(S,\Gamma,\cdot,\le)$ in which the following conditions hold.

(i) $(S,\Gamma,\cdot)$ is an LA-Γ-semigroup.

(ii) $(S,\le)$ is a poset (i.e. reflexive, anti-symmetric and transitive).

(iii) $\forall\ a,b,x \in$ S we have that $a \le b$ implies $a\alpha x \le b\alpha x$ and $x\alpha a \le x\alpha b$, where $\alpha \in \Gamma$.

Example 2.2. Consider $S = \{a,b,c\}$ and $\Gamma = \{\gamma_1,\gamma_2,\gamma_3\}$ with the order "≤" be defined by as per the table given below

$\cdot$	a	b	c
a	a	c	c
b	a	b	b
c	a	b	b

γ_1	a	b	c
a	c	c	c
b	c	c	c
c	c	c	c

γ_2	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

γ_3	a	b	c
a	b	b	b
b	b	b	b
c	b	b	a

We find that $(a\gamma_1 b)\gamma_2 c = (c\gamma_1 b)\gamma_2 a \ \forall \ a,b,c \in S$ and $\forall \ \gamma_1,\gamma_2,\gamma_3 \in \Gamma$ whereas $(a\gamma_1 b)\gamma_2 c \neq a\gamma_1(b\gamma_2 c)$ i.e. associativity is not satisfied. Hence S is an ordered LA-Γ-semiggroup but not an ordered Γ-semiggroup. For the above tables we define left invertive law as per the following order:

$$\leq := \{(a,a),(b,b),(c,c),(a,c),(a,b)\}.$$

For a non-empty subset A of an ordered LA-Γ-semigroup S, we define

$$(A] = \{t \in S \mid t \leq a, \text{for some } a \in A\}.$$

For $A = \{a\}$, we shall write $(a]$.

Definition 2.3. A non-empty subset A of an ordered LA-Γ-semigroup S, is called an LA-Γ-subsemigroup of S if $A\Gamma A \subseteq A$.

Definition 2.4 ([14])**.** A non-empty subset A of an ordered LA-Γ-semigroup S is called a left (right) ideal of S if

(i) $S\Gamma A \subseteq A$ $(A\Gamma S \subseteq A)$.

(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Equivalently, a non-empty subset A of an ordered LA-Γ-semigroup S is called a left (right) Γ-ideal of S if $(S\Gamma A] \subseteq A$ $((A\Gamma S] \subseteq A)$.

A non-empty subset A of an ordered LA-Γ-semigroup S is called a (two sided) Γ-ideal of S if it is both a left and a right Γ-ideal of S.

Definition 2.5. An LA-Γ-subsemigroup A of an ordered LA-Γ semigroup S is called a bi-Γ-ideal of S if

(i) $(A\Gamma S)\Gamma A \subseteq A$.

(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Definition 2.6. An LA-Γ subsemigroup A of an ordered LA-Γ semigroup S is called an interior ideal of S if

(i) $(S\Gamma A)\Gamma S \subseteq A$.

(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Definition 2.7. Let S be an ordered LA-Γ semigroup. A non-empty subset A of S is called a prime if $x\gamma y \in A$ implies $x \in A$ or $y \in A$ for all $x, y \in S$ and $\gamma \in \Gamma$. Let A be an ideal of S. If A is prime subset of S, then A is called prime Γ-ideal.

Definition 2.8. A non-empty subset A of an ordered LA-Γsemigroup S is called a quasi-Γ ideal of S if

(i) $A\Gamma S \cap S\Gamma A \subseteq A$.

(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Definition 2.9. A relation θ on an ordered LA-Γ-semigroup S is called a pseudoorder if

(1) $\leq \subseteq \theta$

(2) θ is transitive, that is $(a,b),(b,c) \in \theta$ implies $(a,c) \in \theta$ for all $a,b,c \in S$.

(3) θ is compatible, that is if $(a,b) \in \theta$ then $(a\gamma x, b\gamma x) \in \theta$ and $(x\gamma a, x\gamma b) \in \theta$ for all $a,b,x \in S$ and $\gamma \in \Gamma$.

An equivalence relation θ on an ordered LA-Γ-semigroup S is called a congruence relation if $(a,b) \in \theta$, then $(a\gamma x, b\gamma x) \in \theta$ and $(x\gamma a, x\gamma b) \in \theta$, for all $a,b,x \in S$ and $\gamma \in \Gamma$. A congruence θ on S is called complete if $[a]_\theta \gamma [b]_\theta = [a\gamma b]_\theta$ for all $a,b \in S$ and $\gamma \in \Gamma$ and $[a]_\theta$ is the congruence class containing the element $a \in S$.

3. Rough Approximations in Ordered LA-Γ-semigroups

Let X be a non-empty set and θ be a binary relation on X. By $\wp(X)$ we mean the power set of X. For all $A \subseteq X$, we define θ_- and $\theta_+ : \wp(X) \to \wp(X)$ by

$$\theta_-(A) = \{x \in X : \forall\ y, x\theta y \Rightarrow y \in A\} = \{x \in X : \theta N(x) \subseteq A\}$$

and

$$\theta_+(A) = \{x \in X : \exists\ y \in A, \text{ such that } x\theta y\} = \{x \in X : \theta N(x) \cap A \neq \emptyset\}.$$

Where $\theta N(x) = \{y \in X : x\theta y\}$. $\theta_-(A)$ and $\theta_+(A)$ are called the lower approximation and the upper approximation operations, respectively [15].

Example 3.1. Let $X = \{1,2,3\}$ and $\theta = \{(1,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$. Then $\theta N(1) = \{1\}$; $\theta N(2) = \{2,3\}$; $\theta N(3) = \{1,2,3\}$; $\theta_-(\{1\}) = \{1\}$; $\theta_-(\{2\}) = \phi$; $\theta_-(\{3\}) = \phi$; $\theta_-(\{1,2\}) = \{1\}$; $\theta_-(\{1,3\}) = \{1\}$; $\theta_-(\{2,3\}) = \{2\}$; $\theta_-(\{1,2,3\}) = \{1,2,3\}$; $\theta_+(\{1\}) = \{1,3\}$; $\theta_+(\{2\}) = \{2,3\}$; $\theta_+(\{3\}) = \{2,3\}$; $\theta_+(\{1,2\}) = \{1,2,3\}$; $\theta_+(\{1,3\}) = \{1,2,3\}$; $\theta_+(\{2,3\}) = \{2,3\}$; $\theta_+(\{1,2,3\}) = \{1,2,3\}$.

Theorem 3.2 ([19])**.** *Let θ and λ be relations on X. If A and B are non-empty subsets of S. Then the following hold:*

(1) $\theta_-(X) = X = \theta_+(X)$;

(2) $\theta_-(\emptyset) = \emptyset = \theta_+(\emptyset)$;

(3) $\theta_-(A) \subseteq A \subseteq \theta_+(A)$;

(4) $\theta_+(A \cup B) = \theta_+(A) \cup \theta_+(B)$;

(5) $\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B)$;

(6) $A \subseteq B$ *implies* $\theta_-(A) \subseteq \theta_-(B)$;

(7) $A \subseteq B$ *implies* $\theta_+(A) \subseteq \theta_+(B)$;

(8) $\theta_-(A \cup B) \supseteq \theta_-(A) \cup \theta_-(B)$;

(9) $\theta_+(A \cap B) \subseteq \theta_+(A) \cap \theta_+(B)$.

Definition 3.3. Let θ be a pseudoorder on an ordered LA-Γ-semigroup S and A be a non-empty subset of S. Then the sets

$$\theta_{-}(A) = \{x \in S : \forall\ y, x\theta y \Rightarrow y \in A\} = \{x \in S : \theta N(x) \subseteq A\}$$

and

$$\theta_{+}(A) = \{x \in S : \exists\ y \in A, \text{ such that } x\theta y\} = \{x \in S : \theta N(x) \cap A \neq \emptyset\}$$

are called the θ-lower approximation and the θ-upper approximation of A.

For a non-empty subset A of S, $\theta(A) = (\theta_{-}(A), \theta_{+}(A))$ is called a rough set with respect to θ if $\theta_{-}(A)$ and $\theta_{+}(A)$ are not same.

Example 3.4. Consider $S = \{1,2,3,4,5\}$ and $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ with the following operation "." and the order "≤":

.	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	5	3	4
5	1	2	3	4	5

γ_1	1	2	3	4	5
1	2	2	2	2	2
2	2	2	2	2	2
3	2	2	2	2	2
4	2	2	2	2	2
5	2	2	2	2	2

γ_2	1	2	3	4	5
1	3	3	3	3	3
2	3	3	3	3	3
3	3	3	3	3	3
4	3	3	3	3	3
5	3	3	3	3	4

γ_3	1	2	3	4	5
1	3	3	3	3	3
2	3	3	3	3	3
3	3	3	3	3	3
4	3	3	3	3	3
5	3	3	3	3	5

$$\leq := \{(1,1),(2,2),(2,3),(2,4),(2,5),(3,3),(4,4),(5,5)\}.$$

We give the covering relation "≺" of S as follows:

$$\prec := \{(2,3),(2,4),(2,5)\}$$

Hence S is an ordered LA-Γ-semigroup because the elements of S satisfies left invertive law.

Now, let

$$\theta = \{(1,1),(1,4),(2,2),(2,3),(2,4),(2,5),(3,3),(4,4),(5,3),(5,4),(5,5)\}$$

be a complete pseudoorder on S, such that

$$\theta N(1) = \{1,4\},\ \theta N(2) = \{2,3,4,5\} \text{ and } \theta N(3) = \{3\},\ \theta N(4) = \{4\},\ \theta N(5) = \{3,4,5\}.$$

Now for $A = \{1,2,4\} \subseteq S$,

$$\theta_-(\{1,2,4\}) = \{1,4\} \text{ and } \theta_+(\{1,2,4\}) = \{1,2,3,4,5\}.$$

So, $\theta_-(\{1,2,4\})$ is θ-lower approximation of A and $\theta_+(\{1,2,4\})$ is θ-upper approximation of A.

For a non-empty subset A of S, $\theta(A) = (\theta_-(A),\theta_+(A))$ is called a rough set with respect to θ if $\theta_-(A) \neq \theta_+(A)$.

Theorem 3.5. *Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. If A and B are non-empty subsets of S. Then $\theta_+(A)\Gamma\theta_+(B) \subseteq \theta_+(A\Gamma B)$.*

Proof. Let z be any element of $\theta_+(A)\Gamma\theta_+(B)$. Then $z = x\gamma y$ where $x \in \theta_+(A)$ and $y \in \theta_+(B)$ and $\gamma \in \Gamma$. Thus there exist elements $c,d \in S$ such that

$$c \in A \text{ and } x\theta c\ ;\ d \in B \text{ and } y\theta m.$$

Since θ is a pseudoorder on S, so $x\gamma y\theta c\gamma d$. As $a\gamma b \in A\Gamma B$, so we have

$$z = x\gamma y \in \theta_+(A\Gamma B).$$

Thus $\theta_+(A)\Gamma\theta_+(B) \subseteq \theta_+(A\Gamma B)$. □

Definition 3.6. Let θ be a pseudoorder on an ordered LA-Γ-semigroup S, then for each $x,y \in S$ and $\gamma \in \Gamma$, $\theta N(x)\Gamma\theta N(y) \subseteq \theta N(x\gamma y)$. If

$$\theta N(x)\Gamma\theta N(y) = \theta N(x\gamma y),$$

then θ is called complete pseudoorder.

Theorem 3.7. *Let θ be a complete pseudoorder on an ordered LA-Γsemigroup S. If A and B are non-empty subsets of S. Then*

$$theta_-(A)\Gamma\theta_-(B) \subseteq \theta_-(A\Gamma B).$$

Proof. Let z be any element of $\theta_-(A)\Gamma\theta_-(B)$. Then $z = x\gamma y$ where $x \in \theta_-(A)$ and $y \in \theta_-(B)$. Thus, we have $\theta N(x) \subseteq A$ and $\theta N(y) \subseteq B$. Since θ is complete pseudoorder on S, so we have

$$\theta N(x\gamma y) = \theta N(x)\Gamma\theta N(y) \subseteq A\Gamma B,$$

which implies that $x\gamma y \in \theta_-(A\Gamma B)$. Thus $\theta_-(A)\Gamma\theta_-(B) \subseteq \theta_-(A\Gamma B)$. □

4. Rough Ideals in Ordered LA-Γ-semigroups

Definition 4.1. Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough LA-Γ-subsemigroup of S if $\theta_+(A)$ (resp., $\theta_-(A)$) is an LA-Γ-subsemigroup of S.

Theorem 4.2. *Let θ be a pseudoorder on an ordered LA-Γ-semigroup S and A be an LA-Γ-subsemigroup of S. Then*

(1) *$\theta_+(A)$ is an LA-Γ-subsemigroup of S.*

(2) *If θ is complete, then $\theta_-(A)$ is, if it is non-empty, an LA-Γ-subsemigroup of S.*

Proof. **(1)** Let A be an LA-Γ-subsemigroup of S. Then by Theorem 3.2(3),

$$\emptyset \neq A \subseteq \theta_+(A).$$

By Theorem 3.2(7) and Theorem 3.5, we have

$$\theta_+(A)\Gamma\theta_+(A) \subseteq \theta_+(A\Gamma A) \subseteq \theta_+(A).$$

Thus $\theta_+(A)$ is an LA-Γ-subsemigroup of S, that is, A is a θ-upper rough LA-Γ-subsemigroup of S.

(2) Let A be an LA-Γ-subsemigroup of S. Then by Theorem 3.2(6) and Theorem 3.7, we have

$$\theta_-(A)\Gamma\theta_-(A) \subseteq \theta_-(A\Gamma A) \subseteq \theta_-(A).$$

Thus $\theta_-(A)$ is, if it is non-empty, an LA-Γ-subsemigroup of S, that is, A is a θ-lower rough LA-Γ-subsemigroup of S. □

The following example shows that the converse of above theorem does not hold.

Example 4.3. We consider a set $S = \{1,2,3,4,5\}$ and $\Gamma = \{\gamma_1,\gamma_2,\gamma_3\}$ with the following operation "." and the order "≤":

.	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	3	4	5
5	1	2	5	3	4

γ_1	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1

γ_2	1	2	3	4	5
1	4	4	4	4	4
2	4	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4
5	4	4	4	4	2

γ_3	1	2	3	4	5
1	3	3	3	3	3
2	3	3	3	3	3
3	3	3	3	3	3
4	3	3	3	3	3
5	3	3	3	3	5

$\leq := \{(1,1),(1,2),(2,2),(2,4),(3,3),(4,4),(5,5)\}$.

We give the covering relation "$\prec$" of S as follows:

$\prec := \{(1,2)\}$

Hence S is an ordered LA-Γ-semigroup because the elements of S satisfies left invertive law.

Here S is not an ordered semigroup because $3 = 3\cdot(4\cdot 5) \neq (3\cdot 4)\cdot 5 = 4$. Hence S is an ordered LA-semigroup because the elements of S satisfies left invertive law.

Now, let

$$\theta = \{1,1),(1,2),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$$

be a complete pseudoorder on S, such that

$$\theta N(1) = \{1,2\},\ \theta N(2) = \{2\} \text{ and } \theta N(3) = \theta N(4) = \theta N(5) = \{3,4,5\}.$$

Now for $\{1,2,3\} \subseteq S$,

$$\theta_-(\{1,2,3\}) = \{1,2\} \text{ and } \theta_+(\{1,2,3\}) = \{1,2,3,4,5\}.$$

It is clear that $\theta_-(\{1,2,3\})$ and $\theta_+(\{1,2,3\})$ are both LA-Γ-subsemigroups of S but $\{1,2,3\}$ is not an LA-Γ-subsemigroup of S.

Definition 4.4. Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough left Γ-ideal of S if $\theta_+(A)$ (resp., $\theta_-(A)$) is a left Γ-ideal of S.

Similarly, we can define θ-upper, θ-lower rough Γ-right ideal and θ-upper, θ-lower rough two-sided Γ-ideals of S.

Theorem 4.5. *Let θ be a pseudoorder on an ordered LA-Γ-semigroup S and A be a left (right, two-sided) Γ-ideal of S. Then*

(1) *$\theta_+(A)$ is a Γ-left (right, two-sided)-ideals of S.*

(2) *If θ is complete, then $\theta_-(A)$ is, if it is non-empty, a left (right, two-sided) Γ-ideal of S.*

Proof. (1) Let A be a left Γ-ideal of S. By Theorem 3.2(1), $\theta_+(S) = S$.

(i) Now by Theorem 3.5, we have

$$S\Gamma\theta_+(A) = \theta_+(S)\Gamma\theta_+(A) \subseteq \theta_+(S\Gamma A) \subseteq \theta_+(A).$$

(ii) Let $a \in \theta_+(A)$ and $b \in S$ such that $b \leq a$. Then there exist $y \in A$, such that $a\theta y$ and $b\theta a$. Since θ is transitive, so $b\theta y$ implies $b \in \theta_+(A)$.

Thus $\theta_+(A)$ is a left Γ-ideal of S, that is, A is a θ-upper rough left Γ-ideal of S.

(2) Let A be a left Γ-ideal of S. By Theorem 3.2(1), $\theta_-(S) = S$.

(i) Now by Theorem 3.7, we have

$$S\Gamma\theta_-(A) = \theta_-(S)\Gamma\theta_-(A) \subseteq \theta_-(S\Gamma A) \subseteq \theta_-(A).$$

(ii) Let $a \in \theta_-(A)$ and $b \in S$ such that $b \le a$. Then $[a]_\theta \subseteq A$ and $b\theta a$. This implies that $[a]_\theta = [b]_\theta$. Since $[a]_\theta \subseteq A$, so $[b]_\theta \subseteq A$. Thus $b \in \theta_-(A)$.

Thus $\theta_-(A)$ is, if it is non-empty, a left Γ-ideal of S, that is, A is a θ-lower rough left Γ-ideal of S. The other cases can be proved in a similar way. □

Definition 4.6. Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough bi-Γ-ideal of S if $\theta_+(A)$ (resp., $\theta_-(A)$) is a bi-Γ-ideal of S.

Theorem 4.7. *Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. If A is a bi-Γ-ideal of S, then it is a θ-upper rough bi-Γ-ideal of S.*

Proof. Let A be a bi-Γ-ideal of S.

(i) By Theorem 3.5, we have

$$(\theta_+(A)\Gamma S)\Gamma\theta_+(A) = (\theta_+(A)\Gamma\theta_+(S))\Gamma\theta_+(A) \subseteq \theta_+((A\Gamma S)A) \subseteq \theta_+(A).$$

(ii) Let $a \in \theta_+(A)$ and $b \in S$ such that $b \le a$. Then there exist $y \in A$, such that $a\theta y$ and $b\theta a$. Since θ is transitive, so $b\theta y$ implies $b \in \theta_+(A)$.

From this and Theorem 4.2(1), we have $\theta_+(A)$ is a bi-Γ-ideal of S, that is, A is a θ-upper rough bi-Γ-ideal of S. □

Theorem 4.8. *Let θ be a complete pseudoorder on an ordered LA-Γ-semigroup S. If A is a bi-Γ-ideal of S, then $\theta_-(A)$ is, if it is non-empty, a bi-Γ-ideal of S.*

Proof. Let A be a bi-Γ-ideal of S.

(i) By Theorem 3.7, we have

$$(\theta_-(A)\Gamma S)\Gamma\theta_-(A) = (\theta_-(A)\Gamma\theta_-(S))\Gamma\theta_-(A) \subseteq \theta_-((A\Gamma S)\Gamma A) \subseteq \theta_-(A).$$

(ii) Let $a \in \theta_-(A)$ and $b \in S$ such that $b \le a$. Then $[a]_\theta \subseteq A$ and $b\theta a$. This implies that $[a]_\theta = [b]_\theta$. Since $[a]_\theta \subseteq A$, so $[b]_\theta \subseteq A$. Thus $b \in \theta_-(A)$.

From this and Theorem 4.2(2), we obtain that $\theta_-(A)$ is, if it is non-empty, a bi-Γ-ideal of S. □

Theorem 4.9. *Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. If A and B are a right and a left Γ-ideal of S respectively, then*

$$\theta_+(A\Gamma B) \subseteq \theta_+(A) \cap \theta_+(B).$$

Proof. The proof is straightforward. □

Theorem 4.10. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A is a right and B is a left ideal of S, then*

$$\theta_{-}(A\Gamma B) \subseteq \theta_{-}(A) \cap \theta_{-}(B).$$

Proof. The proof is straightforward. □

Definition 4.11. Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough interior Γ-ideal of S if $\theta_{+}(A)$ (resp., $\theta_{-}(A)$) is an interior Γ-ideal of S.

Theorem 4.12. *Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. If A is an interior Γ-ideal of S, then A is a θ-upper rough interior Γ-ideal of S.*

Proof. The proof of this theorem is similar to the Theorem 4.7. □

Theorem 4.13. *Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. If A is an interiorΓ-ideal of S, then $\theta_{-}(A)$ is, if it is non-empty, an interior Γ-ideal of S.*

Proof. The proof of this theorem is similar to the Theorem 4.8. □

We call A a rough interior Γ-ideal of S if it is both a θ-lower and θ-upper rough interior Γ-ideal of S.

Definition 4.14. Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. Then a non-empty subset Q of S is called a θ-upper (resp., θ-lower) rough quasi-Γ-ideal of S if $\theta_{+}(Q)$ (resp., $\theta_{-}(Q)$) is a quasi-Γ-ideal of S.

Theorem 4.15. *Let θ be a complete pseudoorder on an ordered LA-Γ-semigroup S. If Q is a quasi-Γ-ideal of S, then Q is a θ-lower rough quasi-Γ-ideal of S.*

Proof. Let Q be a quasi-Γ-ideal of S.

(i) Now by Theorem 3.2(5) and Theorem 3.7, we get

$$\begin{aligned}\theta_{-}(Q)\Gamma S \cap S\Gamma\theta_{-}(Q) &= \theta_{-}(Q)\Gamma\theta_{-}(S) \cap \theta_{-}(S)\Gamma\theta_{-}(Q) \\ &\subseteq \theta_{-}(Q\Gamma S) \cap \theta_{-}(S\Gamma Q) \\ &= \theta_{-}(Q\Gamma S \cap S\Gamma Q) \\ &\subseteq \theta_{-}(Q).\end{aligned}$$

(ii) Let $a \in \theta_{-}(Q)$ and $b \in S$ such that $b \le a$. Then $[a]_\theta \subseteq Q$ and $b\theta a$. This implies that $[a]_\theta = [b]_\theta$. Since $[a]_\theta \subseteq Q$, so $[b]_\theta \subseteq Q$. Thus $b \in \theta_{-}(Q)$.

Thus we obtain that $\theta_{-}(Q)$ is a quasi-Γ-ideal of S, that is, Q is a θ-lower rough quasi-Γ-ideal of S. □

Theorem 4.16. *Let θ be a complete pseudoorder on an ordered LA-Γ-semigroup S. Let L and R be a θ-lower rough left Γ-ideal and a θ-lower rough right Γ-ideal of S, respectively. Then $L \cap R$ is a θ-lower rough quasi-Γ-ideal of S.*

Proof. The proof is straightforward. □

Definition 4.17. Let θ be a pseudoorder on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough prime Γ-ideal of S if $\theta_+(A)$ (resp., $\theta_-(A)$) is a prime Γ-ideal of S.

Theorem 4.18. *Let θ be a complete pseudoorder on an ordered LA-Γ-semigroup S. If A is a prime Γ-ideal of S. Then A is a θ-upper rough prime Γ-ideal of S.*

Proof. Since A is a prime Γ-ideal of S, it follows from Theorem 4.5(1), that $\theta_+(A)$ is an Γ-ideal of S. Let $x\gamma y \in \theta_+(A)$ for some $x, y \in S$ and $\gamma \in \Gamma$. Then

$$\theta N(x\gamma y) \cap A = \theta N(x)\Gamma\theta N(y) \cap A \neq \emptyset,$$

so there exist elements

$$x' \in \theta N(x) \text{ and } y' \in \theta N(y), \text{ such that } x'\gamma y' \in A.$$

Since A is a prime Γ-ideal of S, so we have $x' \in A$ or $y' \in A$. Thus $\theta N(x) \cap A \neq \phi$ or $\theta N(y) \cap A \neq \phi$, and so $x \in \theta_+(A)$ or $y \in \theta_+(A)$. Therefore $\theta_+(A)$ is a prime Γ-ideal of S. □

Theorem 4.19. *Let θ be a complete pseudoorder on an ordered LA-Γ-semigroup S and A be a prime Γ-ideal of S. Then $\theta_-(A)$ is, if it is non-empty, a prime Γ-ideal of S.*

Proof. Since A is an Γ-ideal of S, by Theorem 4.5(2), we have, $\theta_-(A)$ is an Γ-ideal of S. Let

$$x\gamma y \in \theta_-(A) \text{ for some } x, y \in S.$$

Then

$$\theta N(x\gamma y) \subseteq A, \text{ which implies that } \theta N(x)\Gamma\theta N(y) \subseteq \theta N(x\gamma y) \subseteq A.$$

We suppose that $\theta_-(A)$ is not a prime Γ-ideal of S. Then there exists $x, y \in S$ and $\gamma \in \Gamma$ such that $x\gamma y \in \theta_-(A)$ but $x \notin \theta_-(A)$ and $y \notin \theta_-(A)$. Thus $\theta N(x) \nsubseteq A$ and $\theta N(y) \nsubseteq A$. Then there exists $x' \in \theta N(x)$, $x' \notin A$ and $y' \in \theta N(y)$, $y' \notin A$. Thus

$$x'\gamma y' \in \theta N(x)\Gamma\theta N(y) \subseteq A.$$

Since A is a prime Γ-ideal of S, we have $x' \in A$ or $y' \in A$. It contradicts our supposition. This means that $\theta_-(A)$ is, if it is non-empty, a prime Γ-ideal of S. □

We call A a rough prime Γ-ideal of S, if it is both a θ-lower and a θ-upper rough prime Γ-ideal of S.

The following example shows that the converse of Theorem 4.18 and Theorem 4.19 does not hold.

5. Conclusion

The properties of ordered LA-Γ-semigroups in terms of rough sets have been discussed. Then through pseudoorders, it is proved that the lower and upper approximations of two-sided ideals (resp., bi-ideals and prime ideals) in ordered LA-semigroups becomes two-sided ideals (resp., bi-ideals and prime ideals).

In our future studies, following topics may be considered:

1. Rough prime bi-ideals of ordered LA-Γ-semigroups.
2. Rough fuzzy ideals in ordered LA-Γ-semigroups.
3. Rough fuzzy prime bi-ideals of ordered LA-Γ-semigroups.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] M.I. Ali, T. Mahmood and A. Hussain, A study of generalized roughness in $(\in, \in \vee qk)$-fuzzy filters of ordered semigroups, *Journal of Taibah University For Science* **12**(2) (2018), pages 1 – 10, DOI: 10.1080/16583655.2018.1451067.

[2] M.A. Ansari and A.N.A. Koam, Rough approximations in KU-algebras, *Italian J. of Pure and Applied Mathematics* **40** (2018), 679 – 691.

[3] M.A. Ansari and N. Yaqoob, T-rough ideals in ternary semigroups, *International Journal of Pure and Applied Mathematics* **86**(2) (2013), 411 – 424, DOI: 10.12732/ijpam.v86i2.15.

[4] M.A. Ansari, A.N.A. Koam and A. Haider, Rough set theory applied to UP-algebras, *Ital. J. Pure Appl. Math.* (to appear).

[5] M. Aslam, M. Shabir, N. Yaqoob and A. Shabir, On rough (m,n) bi-ideals and generalized rough (m,n) bi-ideals in semigroups, *Ann. Fuzzy Math. Inform.* **2**(2) (2011), 141 – 150.

[6] R. Biswas and S. Nanda, Rough groups and rough subgroups, *Bull. Polish Acad. Sci. Math.* **42** (1994), 251 – 254.

[7] T.K. Dutta and N.C. Adhikari, On prime radicals of Γ-semigroups, *Bull. Calcutta. Math. Soc.* **86**(5) (1994), 437 – 444.

[8] Faisal, M. Khan, B. Davvaz and S. Haq, A note on fuzzy ordered AG-groupoids, *Journal of Intelligent and Fuzzy Systems* **26** (2014), 2251 – 2261, DOI: 10.3233/IFS-130898.

[9] P. Holgate, Groupoids satisfying a simple invertive law, *The Math. Stud.* **61** (1992), 101 – 106.

[10] M.A. Kazim and M. Naseeruddin, On almost semigroups, *Aligarh Bull. Math.* **2** (1972), 1 – 7.

[11] N. Kehayopulu, On ordered Γ-semigroups, *Scientiae Mathematicae Japonicae Online* **e-2010**, 37 – 43.

[12] M. Khan and Faisal, On fuzzy ordered Abel-Grassmann's groupoids, *J. Math. Res.* **3** (2011), 27 – 40, DOI: 10.5539/jmr.v3n2p27.

[13] A. Khan, Faisal, W. Khan and N. Yaqoob, Ordered LA-semigroups in terms of interval valued fuzzy ideals, *Journal of Advanced Research in Pure Mathematics* **5**(1) (2013), 100 – 117.

[14] M. Khan, V. Amjid, G. Zaman and N. Yaqoob, Characterization of ordered Γ abel-grassmann's Groupoids, *Discussiones Mathematicae, General Algebra and Applications* **34** (2014), 55 – 73, DOI: 10.7151/dmgaa.1217.

[15] M. Kondo, On the structure of generalized rough sets, *Inform. Sci.* **176** (2006), 589 – 600, DOI: 10.1016/j.ins.2005.01.001.

[16] N. Kuroki, Rough ideals in semigroups, *Inform. Sci.* **100** (1997), 139 – 163, DOI: 10.1016/S0020-0255(96)00274-5.

[17] Q. Mushtaq and S.M. Yusuf, On LA-semigroup defined by a commutative inverse semigroups, *Math. Bech.* **40** (1988), 59 – 62.

[18] Q. Mushtaq and S.M. Yusuf, On LA-semigroups, *Aligarh Bull. Math.* **8** (1978), 65 – 70.

[19] Z. Pawlak, Rough sets, *Int. J. Comput. Inform. Sci.* **11** (1982), 341 – 356, DOI: 10.1007/BF01001956.

[20] N. Rehman, C. Park, S.I.A. Shah and A. Ali, On generalized roughness in LA-semigroups, *Mathematics* **6** (2018), p. 112, DOI: 10.3390/math6070112.

[21] M.K. Sen and N.K. Saha, On Γ-semigroup-I, *Bull. Calcutta Math. Soc.* **78**(3) (1986), 180 – 186.

[22] M.K. Sen, On Γ-semigroups, *Algebra and its Applications* (New Delhi, 1981), 301-308, in *Lecture Notes in Pure and Appl. Math.* **91**, Dekker, New York (1984).

[23] T. Shah and I. Rehman, On Γ-ideals and bi-Γ-ideals in Γ-AG-groupoid, *Int. J. Algebra* **4** (2010), 267 – 276.

[24] T. Shah, I. Rehman and A. Ali, On ordering of AG-groupoids, *Int. Electronic J. Pure Appl. Math.* **2**(4) (2010), 219 – 224.

[25] Q.M. Xiao and Z.L. Zhang, Rough prime ideals and rough fuzzy prime ideals in semigroups, *Inform. Sci.* **176** (2006), 725 – 733, DOI: 10.1016/j.ins.2004.12.010.

[26] N. Yaqoob and R. Chinram, On prime (m,n) bi-ideals and rough prime (m,n) bi-ideals in semigroups, *Far East J. Math. Sci.* **62**(2) (2012), 145 – 159.

[27] N. Yaqoob, M. Aslam and R. Chinram, Rough prime bi-ideals and rough fuzzy prime bi-ideals in semigroups, *Ann. Fuzzy Math. Inform.* **3**(2) (2012), 203 – 211.