Communications in Mathematics and Applications
Vol. 10, No. 1, pp. 55–69, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i1.1146

Research Article

The Finite Family L-Lipschitzian Suzuki-Generalized Nonexpansive Mappings

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Abstract. In this paper, we propose and analyze a L-Lipschitzian Suzuki-generalized nonexpansive mapping on a nonempty subset of a hyperbolic space and prove Δ-convergence theorems and convergence theorems for a L-Lipschitzian Suzuki-generalized nonexpansive mapping in a hyperbolic space.

Keywords. Fixed point set; L-Lipschitzian Suzuki-generalized nonexpansive mappings; Iteration and hyperbolic spaces

MSC. 47H05; 47H10; 47J25

Received: December 9, 2018 **Accepted:** February 7, 2019

1. Introduction

In mathematics, a fixed point of a function is an element of the function's domain that is mapped to itself by the function. That is to say, x_0 is a fixed point of the function $f(x)$ if $f(x_0) = x_0$. In numerical analysis, fixed-point iteration is a method of computing fixed points of iterated functions. Let M be a nonempty subset of a linear space X, and let $F(T) = \{x \in M : Tx = x\}$ denotes the set of fixed points of the mapping T on M. Many nonlinear equations are naturally formulated as fixed point equations,

$$x = Tx, \tag{1.1}$$

where $T : X \to X$ is a mappings. A solution x of the equation (1.1) is called a *fixed point* of the mapping T. We consider a Picard iteration, which is given by

$$x_{n+1} = Tx_n, \quad \forall\, \mathbb{N}. \tag{1.2}$$

For the Banach contraction mapping theorem, the Picard iteration converges unique fixed point of T, but it fails to approximate fixed point for nonexpansive mappings, even when the existence of a fixed point of T is guaranteed (see [8]). Consider $T : [0,1] \to [0,1]$ defined by $Tx = 1-x$ for $x \in [0,1]$. Then T is nonexpansive with a unique fixed point at $x = \frac{1}{2}$. If we choose a starting value $x = a \neq \frac{1}{2}$, then the successive iteration of T yield the sequence $\{1-a, a, 1-a, \cdots\}$ (see [8]). Next, let (X,d) be metric space and let M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *nonexpansive*, if

$$d(Tx,Ty) \leq d(x,y), \tag{1.3}$$

for each $x, y \in M$. Define a mapping T on $[0,1]$ by

$$Tx = x.$$

It is easy to see that T is nonexpansive. In the last fifty years, the numerous numbers of researchers attracted in these direction and developed iterative process has been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings (see e.g., [2]-[22]), and compare which one is faster to approximate the fixed point as earliest as possible.

Let (X,d) be metric space and let M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *quasi-nonexpansive*, if

$$d(Tx,p) \leq d(x,p)$$

for each $x \in M$ and $p \in F(T)$. Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0, & x \neq 3, \\ 2, & x = 3. \end{cases}$$

Then $F(T) = \{0\} \neq \emptyset$ and T is quasi-nonexpansive, but T does not satisfy condition C (see [25]).

In 2008, Suzuki [25] introduced a class of single valued mappings called Suzuki-generalized nonexpansive mappings (or condition C), as follows: Let T be a self-mapping on a subset M of a metric space X. Then T is said to satisfy *condition C* if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq d(x,y),$$

for each $x, y \in M$.

It is obvious that every nonexpansive mapping satisfies condition C, but the converse is not true, that is condition C is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. The next simple example can show this fact. We see that, if define a mapping

T_1 and T_2 on $[0,3]$ by

$$T_1x = \begin{cases} 0, & x \neq 3, \\ 1, & x = 3 \end{cases}$$

and

$$T_2x = \begin{cases} 0, & x \neq 3, \\ \frac{3}{2}, & x = 3. \end{cases}$$

Then T_1 and T_2 are condition C, but T_1 and T_2 are not nonexpansive (see [25]).

Definition 1.1. Let (X,d) be a metric space and M be its nonempty subset. Then $T : M \to M$ said to be i if there exists a constant $L > 0$ such that

$$d(Tx,Ty) \leq Ld(x,y)$$

for all $x, y \in M$.

Example 1.2. Consider, $T : [0,2] \to [0,2]$, define by

$$Tx = x^2, \quad \forall\, x \in [0,2].$$

It is easy to see that T is L-Lipschitzian, but T is not nonexpansive.

In 2011, Sahu [20] introduced Normal S-iteration Process, whose rate of convergence similar to the Picard iteration process and faster than other fixed point iteration processes, as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, for each $x_1 \in M$, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is real sequences in $(0,1)$.

In 2014, Kadioglu [10] defined Picard normal S-iteration process (PNS) as follows: With C, X and T as in (NS), for each $x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ is real sequences in $(0,1)$. If $\beta_n = 0$ and $\alpha_n = \beta_n = 0$ in (1.5) then it reduces to Normal S-iteration process and Picard iteration process respectively.

On the other hand, Kohlenbach [13] introduced hyperbolic spaces, as follows: A hyperbolic space is a triple (X,d,W), where (X,d) is a metric space and $W : X^2 \times [0,1] \to X$ is such that

W1: $d(u,W(x,y,\alpha)) \leq \alpha d(u,x) + (1-\alpha)d(u,y)$;

W2: $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha - \beta| d(x,y)$;

W3: $W(x,y,\alpha) = W(y,x,1-\alpha)$;

W4: $d(W(x,z,\alpha),W(y,w,\alpha)) \leq (1-\alpha)d(x,y) + \alpha d(z,w)$,

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0,1]$.

Moreover, a metric space is said to be a convex metric space in the sense of Takahashi [26], where a triple (X,d,W) satisfy only $W1$. The concept of hyperbolic spaces in [13] is more restrictive than the hyperbolictype introduced by Goebel and Kirk [6] since $W1$-$W2$ together are equivalent to (X,d,W) being a space of hyperbolic type in [6]. But it is slightly more general than the hyperbolic space defined in Reich and Shafrir [19] (see [13]). This class of metric spaces in [13] covers all normed linear spaces, R-trees in the sense of Tits, the Hilbert ball with the hyperbolic metric (see [7]), Cartesian products of Hilbert balls, Hadamard manifolds (see [19]), and CAT(0) spaces in the sense of Gromov (see [4] for a detailed treatment). A thorough discussion of hyperbolic spaces and a detailed treatment of examples can be found in [13] (see also [6], [7], [19]). Define the function $d : X^2 \to [0,\infty)$ by

$$d(x,y) = \|x-y\|$$

as a metric on X, where X is a real Banach space which is equipped with norm $\|\cdot\|$. Then, we have that (X,d,W) is a hyperbolic space with mapping $W : X^2 \times [0,1] \to X$ defined by $W(x,y,\alpha) = (1-\alpha)x + \alpha y$ (see [24]).

In this paper, we prove some properties of a L-Lipschitzian Suzuki-generalized nonexpansive mapping on a nonempty subset of a hyperbolic space and prove Δ-convergence theorems and convergence theorems for a L-Lipschitzian Suzuki-generalized nonexpansive mapping in a hyperbolic space.

Next, we recall the same basic definitions, notations and some results on hyperbolic spaces that will be used in the later section.

2. Preliminaries

Now, we recall definitions on hyperbolic spaces. If $x,y \to X$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. The following holds even for the more general setting of convex metric space [26], as follows:

$$d(x,W(x,y,\lambda)) = \lambda d(x,y) \quad \text{and} \quad d(y,W(x,y,\lambda)) = (1-\lambda)d(x,y)$$

for all $x,y \in X$ and $\lambda \in [0,1]$.

A hyperbolic space (X,d,W) is uniformly convex [23] if for any $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that for all $a,x,y \in X$,

$$d\left(W\left(x,y,\frac{1}{2}\right),a\right) \le (1-\delta)r$$

provided $d(x,a) \le r, d(y,a) \le r$ and $d(x,y) \ge \varepsilon r$.

A mapping $\eta : (0,1) \times (0,2] \to (0,1]$, which providing such a $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is called as a modulus of uniform convexity [24]. We call the function η is monotone if it decreases with r (for fixed ε), that is, $\eta(r_2,\varepsilon) \le \eta(r_1,\varepsilon)$, $\forall\ r_2 \ge r_1 > 0$.

Let M be a nonempty subset of metric space (X,d) and $\{x_n\}$ be any bounded sequence in X while diam(M) denote the diameter of M. Consider a continuous functional $r_a(\cdot,\{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x), \quad x \in X.$$

The infimum of $r_a(\cdot,\{x_n\})$ over M is said to be the asymptotic radius of $\{x_n\}$ with respect to M and is denoted by $r_a(M,\{x_n\})$. A point $z \in M$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to M if

$$r_a(z,\{x_n\}) = \inf\{r_a(x,\{x_n\}) : x \in M\},$$

the set of all asymptotic centers of $\{x_n\}$ with respect to M is denoted by $AM(M,\{x_n\})$. This set may be empty, a singleton, or certain infinitely many points. If the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r_a(X,\{x_n\}) = r_a(\{x_n\})$ and $AM(X,\{x_n\}) = AM(\{x_n\})$, respectively. We know that for $x \in X, r_a(x,\{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$. It is known that every bounded sequence has a unique asymptotic center with respect to each closed convex subset in uniformly convex Banach spaces and even CAT(0) spaces (see [8]).

Definition 2.1 ([12]). A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$, if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$.

Remark 2.2. We note that Δ-convergence coincides with the usually weak convergence known in Banach spaces with the usual Opial property.

Lemma 2.3 ([15]). *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Lemma 2.4 ([5]). *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_n\}) = \{x_1\}$ and $\{d(x_n,x_1)\}$ converges, then $x = x_1$.*

Lemma 2.5 ([11]). *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n,x) \le c$, $\limsup_{n\to\infty} d(y_n,x) \le c$ and $\lim_{n\to\infty} Wd(x_n,y_n,\alpha_n)$ for some $c \ge 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

Lemma 2.6 ([18]). *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that*

$$\delta_{n+1} \le \beta_n \delta_n + \gamma_n$$

for all $n \in \mathbb{N}$. If $\beta_n \ge 1$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} (\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Definition 2.7 ([21]). Let (X,d) be a metric space and M be it's nonempty subset of X and T be a self-mapping on M, then a sequence $\{x_n\}$ in M is called approximate fixed point sequence for T (AFPS, in short) if $\lim_{n\to\infty} d(x_n,Tx_n) = 0$.

Theorem 2.8 ([1]). *Let M be a nonempty closed convex subset of a complete CAT(0) space X, $T : M \to M$ a nearly asymptotically quasi-nonexpansive mapping with sequence $\{a_n,u_n\}$ such* *that* $\sum_{n=1}^{\infty} a_n < \infty$ *and* $\sum_{n=1}^{\infty} u_n < \infty$. *Assume that* $F(T)$ *is a closed set. Let* $\{x_n\}$ *be a sequence in* M. *Then* $\{x_n\}$ *converges strongly to a fixed point of* T *if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

3. Main Results

In this section, we begin with the definition of L-Lipschitzian Suzuki-generalized nonexpansive mapping.

Definition 3.1. Let T be a self-mapping on a subset M of a metric space X. Then T is said to satisfy *L-Lipschitzian Suzuki-generalized nonexpansive* if there exists a constant $L > 0$ such that

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le Ld(x,y), \quad \forall\ n \ge 1, x,y \in M.$$

Example 3.2. Consider, $T : [0,2] \to [0,2]$, define by

$$Tx = \begin{cases} 0, & x \neq 2, \\ x^2, & x = 2. \end{cases}$$

If $x = 2$ and $y \in (0,1)$, then

$$\frac{1}{2}d(2,T2) = 1 \le d(2,y) \quad \text{and} \quad d(T2,Ty) = 4 \le Ld(2,y).$$

Thus, we see that T is not condition C, because $d(T2,Ty) = 4 > d(2,y)$. In other cases, for any $L \ge \frac{4}{d(2,y)}$, a map T satisfies L-Lipschitzian Suzuki-generalized nonexpansive.

Remark 3.3. We consider Example 3.2, for $x = 2$ and $y \in (0,1)$, if we choose $L < \frac{4}{d(2,y)}$, then we have a map T satisfies L-Lipschitzian Suzuki-generalized nonexpansive, but T is not L-Lipschitzian.

Example 3.4. Consider, $T : [0,2] \to [0,2]$, define by

$$Tx = \begin{cases} x^2, & x \neq 2, \\ 1, & x = 2. \end{cases}$$

If $x = 2$ and $y \in \left(1, \frac{1}{2}\right)$, then

$$\frac{1}{2}d(2,T2) = \frac{1}{2} \le d(x,y) \ \text{ and } \ d(T2,Ty) \le Ld(2,y) \tag{3.1}$$

hold. Thus, T satisfies L-Lipschitzian Suzuki-generalized nonexpansive mapping for any $L \ge \frac{d(T2,Ty)}{d(2,y)}$, but T is not condition (C) because $d(T2,Ty) > d(2,y)$. In other cases, it's easy to see that, a map T satisfies L-Lipschitzian Suzuki-generalized nonexpansive.

Remark 3.5. We will see that in Example 3.4, for $x = 2$ and $y \in \left(1, \frac{1}{2}\right)$, if we choose $L < \frac{d(Tx,Ty)}{d(x,y)}$, then we have a map T satisfies L-Lipschitzian Suzuki-generalized nonexpansive, but T is not L-Lipschitzian.

Proposition 3.6. *Let* $\{T_i\}_{i=1}^{k}$ *be a self finite family of* L_i*-Lipschitzian Suzuki-generalized* *nonexpansive mappings on M. Then*

$$d(x_n,T_iy) \le (1+2L_i)d(x_n,T_ix_n)+L_id(x_n,y)$$

for all $x,y \in M$, $\{x_n\}$ is approximate fixed point sequence in M.

Proof. Let $x,y \in M$, since $\{T_i\}_{i=1}^k$ is a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, we have

$$\frac{1}{2}d(x_n,T_ix_n)=0 \le d(x_n,y),$$

for all $n \in \mathbb{N}$, then

$$d(T_ix_n,T_iy) \le L_id(x_n,y).$$

Now, we consider

$$\begin{aligned}d(x_n,T_iy) &\le d(x_n,T_ix_n)+d(T_ix_n,T_i^2x_n)+d(T_i^2x_n,T_iy)\\ &\le (1+L_i)d(x_n,T_ix_n)+L_id(T_ix_n,y)\\ &\le (1+2L_i)d(x_n,T_ix_n)+L_id(x_n,y).\end{aligned}$$

Hence, $d(x_n,T_iy) \le (1+2L_i)d(x_n,T_ix_n)+L_id(x_n,y)$. □

Let (X,d) be a metric space and let M be a nonempty subset of X. We will denote the fixed point set of mapping $\{T_i\}_{i=1}^k$ by $F(T) := \bigcap_{i=1}^{k} F(T_i)$.

Lemma 3.7. *Let M be a nonempty and convex subset of a strictly convex hyperbolic space X. If $\{T_i\}_{i=1}^k$ be a self finite family of u_nL_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, that is there exist a sequence $\{u_n\}$ and $L_i > 0$ such that*

$$\frac{1}{2}d(x,T_ix) \le d(x,y) \Rightarrow d(T_ix,T_iy) \le u_nL_id(x,y),$$

$\forall\ n \ge 1$, $x,y \in M$ with $u_nL_i \to 1$, for all $i=1,2,\dots,k$ and $F(T) \neq \emptyset$. If $\{x_n\},\{y_n\}$ are bounded approximate fixed point sequence in M, then $F(T)$ is closed and convex.

Proof. Assume that $\{x_n\}$ is a sequence in $F(T)$ which converges to some $y \in M$. To show that $y \in F(T)$ by Proposition 3.6, we obtain that

$$\begin{aligned}d(x_n,T_iy) &\le (1+2L_i)d(x_n,T_ix_n)+u_nL_id(x_n,y)\\ &\le (1+2L_i)d(x_n,T_ix_n)+d(x_n,y).\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} d(x_n,T_iy) \le \limsup_{n\to\infty}(1+2L_i)d(x_n,T_ix_n)+\limsup_{n\to\infty} d(x_n,y).$$

Since $\{x_n\} \subseteq F(T)$, we have $\limsup_{n\to\infty} d(x_n,T_iy) \le \limsup_{n\to\infty} d(x_n,y)$. By the uniqueness of the limit point we obtain that $T_iy=y$, that is $y \in F(T)$, and then $F(T)$ is closed.

Now, we will to show that $F(T)$ is convex. Let $x,y \in F(T)$ and each $\alpha \in (0,1)$. Then,

$$\begin{aligned}d(x,y) &\le d(x,T_i(W(x,y,\alpha)))+d(T_i(W(x,y,\alpha)),y)\\ &\le d(x,W(x,y,\alpha))+d(W(x,y,\alpha),y)\end{aligned}$$

$$\le d(x,y).$$

Now, we consider

$$\begin{aligned} d(x,T_i(W(x,y,\alpha))) &\le (1+2L_i)d(x,T_ix)+u_nL_id(x,W(x,y,\alpha)) \\ &\le (1+2L_i)d(x,T_ix)+d(x,W(x,y,\alpha)) \\ &\le d(x,W(x,y,\alpha)) \end{aligned}$$

and

$$\begin{aligned} d(y,T_i(W(x,y,\alpha))) &\le (1+2L_i)d(y,T_iy)+u_nL_id(y,W(x,y,\alpha)) \\ &\le (1+2L_i)d(y,T_iy)+d(y,W(x,y,\alpha)) \\ &\le d(y,W(x,y,\alpha)), \end{aligned}$$

we get that $d(x,T_i(W(x,y,\alpha))) = d(x,(W(x,y,\alpha)))$ and $d(T_i(W(x,y,\alpha)),y) = d(W(x,y,\alpha),y)$, because if $d(x,T_i(W(x,y,\alpha))) \le d(x,W(x,y,\alpha))$ or $d(T_i(W(x,y,\alpha)),y) \le d(W(x,y,\alpha),y)$, then which the contradiction to $d(x,y) < d(x,y)$. Since M is strictly convex, we have $T_i(W(x,y,\alpha)) = W(x,y,\alpha)$, so $W(x,y,\alpha) \in F(T)$. Hence, $F(T)$ is convex. □

Lemma 3.8. *Let (X,d) be complete uniformly convex hyperbolic space with monotone modulus of convexity η, M be a nonempty closed convex subset of X and $\{T_i\}_{i=1}^k$ be a self finite family of u_nL_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M. Suppose $\{x_n\}$ is bounded sequence in M with $\{x_n\}$ is bounded approximate fixed point sequence for $\{T_i\}_{i=1}^k$, then $\{T_i\}_{i=1}^k$ have a fixed point.*

Proof. Since $\{x_n\}$ is bounded sequence in X, then by Lemma 2.3, has unique asymptotic center in M, that is, $AM(M,\{x_n\}) = \{x\}$ is singleton and $\lim_{n\to\infty} d(x_n,T_ix_n) = 0$. Since $\{T_i\}_{i=1}^k$ satisfies a self finite family of u_nL_i-Lipschitzian Suzuki-generalized nonexpansive on M, there exist a sequence $\{u_n\}$ and $L_i > 0$ such that

$$\begin{aligned} d(x_n,T_ix) &\le (1+2L_i)d(x_n,T_ix_n)+u_nL_id(x_n,x) \\ &\le (1+2L_i)d(x_n,T_ix_n)+d(x_n,x). \end{aligned}$$

Taking lim sup as $n\to\infty$ both the sides, we have

$$\begin{aligned} r_a(T_ix,\{x_n\}) &= \limsup_{n\to\infty} d(x_n,T_ix) \\ &\le \limsup_{n\to\infty}[(1+2L_i)d(x_n,T_ix_n)+d(x_n,x)] \\ &\le \limsup_{n\to\infty} d(x_n,x) = r_a(x,\{x_n\}). \end{aligned}$$

By the uniqueness of asymptotic center, $T_ix = x$, thus x is fixed point of T. Hence, $F(T)$ is nonempty and then $\{T_i\}_{i=1}^k$ has a fixed point. □

Now, we expand the result of Kadioglu [10] (PNS) to L-Lipschitzian Suzuki-generalized nonexpansive mappings in hyperbolic spaces, as follows: Let M be a nonempty closed convex subset of a hyperbolic space X and $\{T_i\}_{i=1}^k$ be a self finite family of L-Lipschitzian Suzuki-generalized nonexpansive mappings on M. For any $x_1 \in M$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = T_i y \\ y_n = W(z_n, T_i z_n, \alpha_n) \\ z_n = W(x_n, T_i x_n, \beta_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0,1]$ for all $n \in \mathbb{N}$.

Theorem 3.9. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let a slef-map $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in* (3.2)*, Δ-converges to a common fixed point of $\{T_i\}_{i=1}^k$.*

Proof. We divide our proof into three steps.
First, we will show that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$. Since $\{T_i\}_{i=1}^k$ satisfies the L_i-Lipschitzian Suzuki-generalized and $p \in F(T)$, we have

$$\frac{1}{2} d(p, T_i p) = 0 \leq d(p, z_n),$$

$$\frac{1}{2} d(p, T_i p) = 0 \leq d(p, y_n)$$

and

$$\frac{1}{2} d(p, T_i p) = 0 \leq d(p, x_n),$$

for all $n \in \mathbb{N}$, we get that

$$d(T_i p, T_i z_n) \leq L_i d(p, z_n),$$

$$d(T_i p, T_i y_n) \leq L_i d(p, y_n)$$

and

$$d(T_i p, T_i x_n) \leq L_i d(p, x_n).$$

By (3.2), we have

$$\begin{aligned} d(z_n, p) &= d(W(x_n, T_i x_n, \beta_n), p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(T_i x_n, p) \\ &= (1-\beta_n) d(x_n, p) + \beta_n d(T_i x_n, T_i p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n L_i d(x_n, p) \\ &= (1-\beta_n + \beta_n L_i) d(x_n, p), \quad \forall\ n \in \mathbb{N}. \end{aligned} \tag{3.3}$$

Using (3.2) and (3.3), we have

$$\begin{aligned} d(y_n, p) &= d(W(z_n, T_i z_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(z_n, p) + \alpha_n d(T_i z_n, p) \\ &= (1-\alpha_n) d(z_n, p) + \alpha_n d(T_i z_n, T_i p) \\ &\leq (1-\alpha_n) d(z_n, p) + \alpha_n L_i d(z_n, p) \end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n+\alpha_n L_i)d(z_n,p)\\
&\le (1-\alpha_n+\alpha_n L_i)[(1-\beta_n+\beta_n L_i)d(x_n,p)]\\
&= [(1-\alpha_n+\alpha_n L_i)(1-\beta_n+\beta_n L_i)]d(x_n,p), \quad \forall\ n\in\mathbb{N}. \qquad (3.4)
\end{aligned}
$$

From (3.3) and (3.4), we have

$$
\begin{aligned}
d(x_{n+1},p) &= d(T_i y_n,p)\\
&= d(T_i y_n, T_i p)\\
&\le L_i d(y_n,p)\\
&\le L_i[(1-\alpha_n+\alpha_n L_i)(1-\beta_n+\beta_n L_i)]d(x_n,p)\\
&= [(L_i-\alpha_n L_i-\beta_n L_i)+(\alpha_n L_i^2+\beta_n L_i^2)+(\alpha_n\beta_n L_i^3-2\alpha_n\beta_n L_i^2+\alpha_n\beta_n L_i)]d(x_n,p)\\
&= \mu_n d(x_n,p), \quad \forall\ n\in\mathbb{N} \qquad (3.5)
\end{aligned}
$$

where $\mu_n=(L_i-\alpha_n L_i-\beta_n L_i)+(\alpha_n L_i^2+\beta_n L_i^2)+(\alpha_n\beta_n L_i^3-2\alpha_n\beta_n L_i^2+\alpha_n\beta_n L_i)$. Since $\sum_{n=1}^{\infty}\alpha_n<\infty$ and $\sum_{n=1}^{\infty}\beta_n<\infty$, so that $\sum_{n=1}^{\infty}(\mu_n-1)<\infty$. Therefore, by Lemma 2.6, we have that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F(T)$.

Secondary step, we prove that $\lim_{n\to\infty} d(x_n,T_i x_n)=0$. Let $\lim_{n\to\infty} d(x_n,p)=c\ge 0$.

(i) If $c=0$, we obviously have

$$
\begin{aligned}
d(x_n,T_i x_n) &\le d(x_n,p)+d(T_i x_n,p)\\
&\le (1+L_i)d(x_n,p),
\end{aligned}
$$

taking lim as $n\to\infty$ on both the sides, we have $\lim_{n\to\infty} d(x_n,T_i x_n)=0$.

(ii) If $c>0$, since $\{T_i\}_{i=1}^k$ is a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings and $p\in F(T)$, we have

$$d(T_i x_n,p)\le L_i d(x_n,p),$$

taking lim sup as $n\to\infty$ both the sides, we have

$$\limsup_{n\to\infty} d(T_i x_n,p)\le L_i c,$$

taking lim sup as $n\to\infty$ both the sides in (3.3), we have

$$\limsup_{n\to\infty} d(z_n,p)\le L_i c. \qquad (3.6)$$

Since

$$d(x_{n+1},p)\le L_i(1-\alpha_n+\alpha_n L_i)d(z_n,p),$$

so, we take lim inf as $n\to\infty$ both the sides, we get

$$
\begin{aligned}
\liminf_{n\to\infty} d(x_{n+1},p) &\le \liminf_{n\to\infty} d(z_n,p)\\
L_i c &\le \liminf_{n\to\infty} d(z_n,p). \qquad (3.7)
\end{aligned}
$$

By (3.6) and (3.7), we have

$$\lim_{n\to\infty} d(z_n,p)=L_i c,$$

it implies that

$$\begin{aligned}L_i c &= \limsup_{n\to\infty} d(z_n,p)\\ &= \limsup_{n\to\infty}[d(W(x_n,T_ix_n,\beta_n),p)]\\ &= \limsup_{n\to\infty}[d((1-\beta_n)x_n \oplus \beta_n T_i x_n,p)]\\ &\le \limsup_{n\to\infty}[(1-\beta_n)d(x_n,p)+\beta_n d(T_ix_n,p)]\\ &\le \limsup_{n\to\infty}(1-\beta_n)d(x_n,p)+\limsup_{n\to\infty}\beta_n d(T_ix_n,p)=L_ic.\end{aligned}$$

From Lemma 2.5, we have $\lim_{n\to\infty} d(x_n,T_ix_n)=0$.

Finally, we will prove that the sequence $\{x_n\}$ Δ-converges to a fixed point of T_i. Since $\{d(x_n,p)\}$ is bounded, by Lemma 2.3, it follows that $\{x_n\}$ has a unique asyptotic center. Let u,v Δ-limits of the subsequence of $\{u_n\},\{v_n\}\subset\{x_n\}$. Since $F(T)\neq\emptyset$, we have u and v are fixed points of $\{T_i\}_{i=1}^k$. Now, we claim that $u=v$. Let $u\neq v$, then by uniqueness of asymptotic center

$$\begin{aligned}\limsup_{n\to\infty} d(x_n,u) &= \limsup_{n\to\infty} d(u_n,u)\\ &< \limsup_{n\to\infty} d(u_n,v)\\ &= \limsup_{n\to\infty} d(x_n,v)\\ &= \limsup_{n\to\infty} d(v_n,v)\\ &< \limsup_{n\to\infty} d(v_n,u)\\ &= \limsup_{n\to\infty} d(x_n,u),\end{aligned}$$

which is a contradiction. Therefore $u=v$, the sequence $\{x_n\}$ Δ-converges to a fixed point of T. This completes the proof. □

Theorem 3.10. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let a self-map $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, such that $F(T)\neq\emptyset$ and $F(T)$ is closed. Then the sequence $\{x_n\}$ defined in* (3.2) *converges strongly to $p\in F(T)$ if and only if $\liminf_{n\to\infty} d(x_n,F(T))=0$, where $d(x_n,F(T))=\inf_{p\in F(T)} d(x_n,p)$.*

Proof. Necessity is obvious, we only prove the sufficiency. Assume that

$$\liminf_{n\to\infty} d(x_n,F(T))=0.$$

From (3.5)

$$d(x_{n+1},F(T))\le \mu_n d(x_n,F(T)),\quad n\in\mathbb{N}$$

then $\lim_{n\to\infty} d(x_n,F(T))$ exists. Hence by the hypothesis, $\liminf_{n\to\infty} d(x_n,F(T))=0$, then we have $\lim_{n\to\infty} d(x_n,F(T))=0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. By Theorem 2.8, we obtained the following inequality

$$d(x_{n+m},p) \le Kd(x_n,p)$$

for each $p \in F(T)$ and for all $m,n \in \mathbb{N}$, where $K = e^{\left(\sum_{j=n}^{n+m-1} \mu_j\right)} > 0$. As, $\sum_{n=1}^{\infty} \mu_n < \infty$ thus $K^* = e^{\left(\sum_{n=1}^{\infty} \mu_n\right)} \ge K = e^{\left(\sum_{j=n}^{n+m-1} \mu_j\right)} > 0$. Let $\epsilon > 0$ be arbitrarily. Since $\lim_{n\to\infty} d(x_n,F(T)) = 0$, there exists a positive integer n_0 such that

$$d(x_n,F(T)) < \frac{\epsilon}{4K^*}, \quad \forall\, n \ge n_0.$$

In particular, $\inf\{d(x_{n_0},p) : p \in F(T)\} < \frac{\epsilon}{4K^*}$. So there exist $p^* \in F(T)$ such that

$$d(x_{n_0},p^*) < \frac{\epsilon}{2K^*}.$$

Thus, for $n \ge n_0$, we have

$$\begin{aligned} d(x_{n+m},x_n) &\le d(x_{n+m},p^*) + d(p^*,x_n) \\ &\le 2K^* d(x_{n_0},p^*) \\ &< 2K^*\left(\frac{\epsilon}{2K^*}\right) = \epsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in M. Since M is a closed subset of a complete uniformly convex hyperbolic space, so it must converge strongly to a point p in M. Since $F(T)$ is closed, $\lim_{n\to\infty} d(x_n,F(T)) = 0$, that is, $p \in F(T)$. This completes the proof. □

4. Conclusion

In this paper, we introduce an algorithm by the iteration process of Kadioglu (PNS) to approximating a fixed point for L-Lipschitzian Suzuki-generalized nonexpansive mappings in hyperbolic spaces and introduce a L-Lipschitzian Suzuki-generalized nonexpansive mapping, i.e.,

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le Ld(x,y).$$

We obtain fixed point theorems, Δ-convergence theorems, and convergence theorems for L-Lipschitzian Suzuki-generalized nonexpansive mappings in a hyperbolic space. Moreover, we obtain that examples, lemmas and theorems for L-Lipschitzian Suzuki-generalized nonexpansive mappings on a nonempty subset of a hyperbolic spaces in the following way:

(1) Let $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M. Then

$$d(x_n,T_iy) \le (1+2L_i)d(x_n,T_ix_n) + L_id(x_n,y)$$

for all $x,y \in M$, $\{x_n\}$ is approximate fixed point sequence in M.

(2) Let M be a nonempty and convex subset of a strictly convex hyperbolic space X. If $\{T_i\}_{i=1}^k$ be a self finite family of u_nL_i-Lipschitzian Suzuki-generalized nonexpansive mappings

on M, that is there exist a sequence $\{u_n\}$ and $L_i > 0$ such that

$$\frac{1}{2}d(x,T_i x) \le d(x,y) \Rightarrow d(T_i x, T_i y) \le u_n L_i d(x,y), \quad \forall\ n \ge 1, x, y \in M.$$

with $u_n L_i \to 1$, for all $i = 1,2,\dots,k$ and $F(T) \neq \emptyset$. If $\{x_n\}$, $\{y_n\}$ are bounded approximate fixed point sequence in M, then $F(T)$ is closed and convex.

(3) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let a slef-map $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (3.2), Δ-converges to a common fixed point of $\{T_i\}_{i=1}^k$.

(4) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let a slef-map $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, such that $F(T) \neq \emptyset$ and $F(T)$ is closed. Then the sequence $\{x_n\}$ defined in (3.2) converges strongly to $p \in F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{p\in F(T)} d(x_n, p)$.

Acknowledgements

The author would like to thank the Rajamangala University of Technology Thanyaburi for financial support.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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