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Research Article

Mixed Energy of a Mixed Hourglass Graph

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Abstract. In this paper we discuss a complete mixed graph called mixed hourglass graph. The direct representation of hourglass matrix in graph gives a weighted mixed hourglass graph. Then, we obtain a mixed hourglass graph from the weighted mixed hourglass graph by assigning its edge-labelled a numerical value of weight 1. Next, we derive the determinant, spectrum and mixed energy of the graph to conclude that the energy of a mixed hourglass graph coincides with twice the number of edges in the graph and the sum of the square of its eigenvalues.

Keywords. Hourglass matrix; Adjacency matrix; Mixed graph; Mixed energy

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1. Introduction

A simple graph $G=(V,E)$ is an ordered pair consisting of a set of vertices $V=\{v_1,v_2,\dots,v_n\}$ and a set of undirected edges $E=\{e_1,e_2,\dots,e_n\}$ [17]. An (unweighted) graph can be considered as a weighted graph (or edge-labelled graph) with each of the edges bearing a numerical value of weight 1 [13]. However, a weighted graph can be a simple graph or a mixed graph. A mixed graph $G=(V,E,A)$ is an ordered triple consisting of a set of vertices V, a set of undirected edges E and a set of directed arcs A [10]. It is important to know that multiple edges are allowed and that a directed loop is considered as an arc in the mixed graph, otherwise when stated. A mixed graph is also the orientation of subset of the undirected graph and its edge set is the union between the set of arcs and the set of undirected edges [19]. Mixed adjacency matrix $M(G)=[m_{ij}]$ of a mixed graph G is defined as an $n\times n$ matrix indexed by the vertices $(v_1,\dots,v_n)$, where $m_{ij}=1$ if $v_iv_j\in E$, $m_{ij}=-1$ if $v_iv_j\in A$, and $m_{ij}=0$ otherwise [5]. A mixed graph is said to be a mixed complete graph if its underlying graph is a simple complete graph, see [14, 18].

The graph energy of a simple graph has been known for many years with its concept originated from theoretical chemistry. It was Ivan Gutman [11] who reintroduced the energy of simple (undirected) graph energy when analyzing π-electron energy using *Huckel molecular orbital* (HMO) theory, where the Hamiltonian operator corresponded to the adjacency matrix. Though, not always that the theory of graph energy coincides with the total π-electron energy [20]. Thus, the graph energy is defined for all graphs and mathematicians can study it without being restricted to any chemistry limitation [12]. The mixed energy $E_M(G)$ of a mixed graph G (likewise for an undirected graph) is the sum of absolute values of eigenvalues, $\lambda_i(G)$, of the adjacency matrix of the graph [11]. A singular graph is a graph with at least one zero eigenvalue, otherwise it is nonsingular [3, 7].

Furthermore, an hourglass graph (or butterfly graph) is a planar undirected graph structured by at least two triangles intersecting in a single vertex, usually from friendship graph F_2 or from 5-vertex graph of two k_3's [2, 15, 16]. Nevertheless, the type of hourglass graph examine in this paper is a complete mixed graph obtained from hourglass matrix. An hourglass matrix (or butterfly matrix) is obtained from quadrant interlocking factorization (or WZ factorization) of nonsingular matrix, see for examples [6, 8, 9]. An hourglass matrix is a nonsingular matrix of order $n\,(n\geq 3)$ with nonzero entries from the ith to the $(n-i+1)$ element of the ith and $(n-i+1)$ row of the matrix, 0's otherwise for $i=1,2,\dots,\lfloor\frac{n+1}{2}\rfloor$ [4]. A direct representation of hourglass matrix in graph gives a weighted complete mixed graph called mixed hourglass graph. In Section 2, we obtained a mixed hourglass graph from a weighted mixed hourglass graph. Then, we examine the determinant, spectrum and mixed energy of a mixed hourglass graph and show that its mixed energy coincides with twice the number of edges in the graph and the sum of the square of its eigenvalues.

2. Mixed Hourglass Graph and Its Mixed Energy

A direct representation of hourglass matrix to weighted mixed hourglass-adjacency matrix will produce a weighted mixed hourglass graph with loops and with/without multiple arcs and undirected edges. There are certain conditions to be met if the weighted mixed hourglass graph of weighted mixed hourglass-adjacency matrix is to be represented, such as taking absolute value of negative weights and/or making all entries on the anti-diagonal the same to avoid multiple arcs. Since there is inconsistence in the representation of weighted mixed hourglass graph from its weighted mixed hourglass-adjacency matrix, we then consider a mixed hourglass-adjacency matrix from the weighted mixed hourglass-adjacency matrix. We do this by replacing the weights (nonzero entries) of the weighted mixed hourglass-adjacency matrix with 1's if there exists an undirected edge, -1's if there exists an arc or loop and 0's otherwise. In order to avoid loops, we assign 0's to the diagonal of the mixed hourglass-adjacency matrix.

In other words, if v_i, v_j is an arc then $h_{i,j} = -1$, if $v_i v_j$ is an undirected edge then $h_{i,j} = h_{j,i} = 1$ (for $i \neq j$, $j = n-1+i$; $i = 1, 2, \dots, n$) and $h_{i,j} = 0$ otherwise. We shall refer a mixed hourglass graph without loops as mixed hourglass graph denoted as $\mathscr{G}$ and its mixed hourglass-adjacency matrix denoted as $M(\mathscr{G})$.

Definition 2.1. A mixed hourglass-adjacency matrix $M(\mathscr{G})$ of a mixed hourglass graph $\mathscr{G}$ is the $n \times n$ matrix $M(\mathscr{G})_{n\times n} = (h_{i,j})_{n\times n}$ defined by

$$M(\mathscr{G}) = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge;} \\ -1 & \text{if } v_i, v_j \text{ is an arc;} \\ 0 & \text{otherwise.} \end{cases}$$

For $n \geq 3$; $i, j = 1, \dots, n$.

Lemma 2.1. *The number of undirected edges e in a mixed hourglass graph $\mathscr{G}$ is*

$$e = \frac{n-\gamma}{2},$$

where $\gamma = \left|(n+1) \bmod 2 - 1\right|$.

Proof. Let e and $d_{\mathscr{G}}(v)$ be the number of undirected edges and the degree of the vertices in $\mathscr{G}$ respectively. Also, let $\phi(i) = n+1-i$, where n is the order of the mixed hourglass graph and $i = 1, 2, \dots, \lceil \frac{n-1}{2} \rceil$. Then, there are pair of vertices with an undirected edge in mixed hourglass graph $\mathscr{G}$ which exists between v_i and $v_{\phi(i)}$. If n is odd, there exists a vertex $v_{\frac{n+1}{2}}$ with no edge but only arcs. Since, the sum of degree of vertices having an edge in $\mathscr{G}$ depends on the order of the graph. If n is odd the sum of degree of vertices is $n-1$ and n if it is even. If we let

$$\sum_{v\in V}^{n} d_{\mathscr{G}}(v) = n - \gamma. \tag{2.1}$$

Such that

$$\gamma = \left|(n+1) \bmod 2 - 1\right| = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

But according to handshaking theorem, the sum of degree of vertices having an edge is twice the number of edges in $\mathscr{G}$. Then

$$\sum_{v\in V}^{n} d_{\mathscr{G}}(v) = 2e. \tag{2.2}$$

Now, substitute Equation (2.1) in Equation (2.2) to have

$$2e = n - \gamma, \tag{2.3}$$

$$n = \frac{n-\gamma}{2}.$$

□

To begin with the mixed energy of a mixed hourglass graph $\mathscr{G}$, we give three examples of mixed hourglass graph of order 4, 5 and 6 in Figure 1 with their determinants as 1, 0 and −1, and their mixed energies as 4, 4 and 6, respectively.

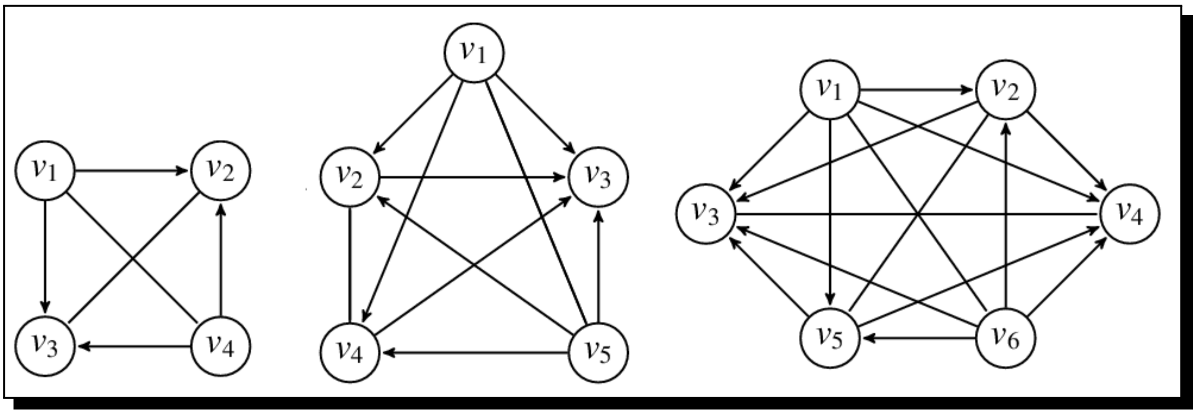

Figure 1. Mixed hourglass graph of order 4, 5 and 6, respectively

Proposition 2.2. *Let $\mathscr{G}$ be a mixed hourglass graph and* $\det(M(\mathscr{G}))$ *the determinant of mixed hourglass-adjacency matrix $M(\mathscr{G})$ of order n. Then*

$$\det(M(\mathscr{G})) = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ -1 & \text{if } n = 2k \text{ where } k \text{ is odd;} \\ 1 & \text{if } n = 2k \text{ where } k \text{ is even.} \end{cases}$$

Proof. Though other methods to prove the proposition exist, we will focus on a method using modulus. Assuming $det(M(\mathscr{G})) = |n \bmod 4 - 2| - 1$. If n is odd, then $n \bmod 4$ produces two values that are relatively prime with 4. They are 1 and 3, respectively.
Thus,

$$1 \le n \bmod 4 \le 3$$
$$|1-2| \le |n \bmod 4 - 2| \le |3-2|$$
$$1-1 \le |n \bmod 4 - 2| - 1 \le 1-1$$
$$|n \bmod 4 - 2| - 1 = 0$$

If n is even, then 0 and 2 are values gotten from $n \bmod 4$. That is,

$$0 \le n \bmod 4 \le 2$$
$$|0-2| \le |n \bmod 4 - 2| \le |2-2|$$
$$0 \le |n \bmod 4 - 2| \le 2$$
$$0-1 \le |n \bmod 4 - 2| - 1 \le 2-1$$

This means if $|n \bmod 4 - 2| - 1 = -1$ then $n = 2k$, where k is odd, otherwise k is even if $|n \bmod 4 - 2| - 1 = 1$.

Therefore,

$$\det(M(\mathscr{G})) = |n \bmod 4 - 2| - 1 = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ -1 & \text{if } n = 2k \text{ where } k \text{ is odd;} \\ 1 & \text{if } n = 2k \text{ where } k \text{ is even.} \end{cases} \quad \square$$

Let the characteristic polynomial, eigenvalues and mixed energy of a mixed hourglass-adjacency matrix $M(\mathscr{G})$ be denoted as $P(\mathscr{G},\lambda)$, $\lambda_i(\mathscr{G})$ and $\mathscr{E}_M(\mathscr{G})$, respectively. Since $\mathscr{G}$ has an underlying undirected graph G which is a complete graph. Whenever $G=\mathscr{G}$ then $M(G)=M(\mathscr{G})$, $P(G,\lambda)=P(\mathscr{G},\lambda)$ and $\mathscr{E}_M(G)=\mathscr{E}_M(\mathscr{G})$. Then

$$P(\mathscr{G},\lambda)=det\left(\lambda\mathbb{I}-M(\mathscr{G})\right).$$

and

$$\mathscr{E}_M(\mathscr{G})=\sum_{i=1}^{n}|\lambda_i(\mathscr{G})|.$$

Theorem 2.3. *Let $M(\mathscr{G})$ be a mixed hourglass-adjacency matrix of a mixed hourglass graph $\mathscr{G}$ and $S_{M(\mathscr{G})}$ the spectrum of mixed hourglass-adjacency matrix. Then*

$$S_{M(\mathscr{G})}=\begin{cases}-1^{(\frac{n}{2})},1^{(\frac{n}{2})} & \textit{if } n \textit{ is even};\\ -1^{(\frac{n-1}{2})},0,1^{(\frac{n-1}{2})} & \textit{if } n \textit{ is odd}.\end{cases}$$

Proof. The eigenvalues of a mixed hourglass-adjacency matrix can be obtained from products of the determinants of its characteristic polynomial via filanz submatrix, see the computation of determinant of hourglass matrix from filanz submatrix in [4]. Then,

$$\det\left(\lambda\mathbb{I}-M(\mathscr{G})\right)=\begin{cases}\prod^{\lceil\frac{n-1}{2}\rceil}\begin{vmatrix}\lambda & -1\\ -1 & \lambda\end{vmatrix} & \text{if } n \text{ is even}\\ |\lambda|\prod^{\lceil\frac{n-1}{2}\rceil}\begin{vmatrix}\lambda & -1\\ -1 & \lambda\end{vmatrix} & \text{if } n \text{ is odd.}\end{cases}$$

Obviously, when n is even we have $\det\left(\lambda\mathbb{I}-M(\mathscr{G})\right)=\prod^{\lceil\frac{n-1}{2}}\rceil(\lambda^2-1)=(\lambda^2-1)^{\lceil\frac{n-1}{2}\rceil}$ and $\det(\lambda\mathbb{I}-M(\mathscr{G})=\lambda\prod^{\lceil\frac{n-1}{2}\rceil}(\lambda^2-1)=\lambda(\lambda^2-1)^{\lceil\frac{n-1}{2}\rceil}$ when n is odd. However, $\lceil\frac{n-1}{2}\rceil=\frac{n}{2}$ when n is even and $\lceil\frac{n-1}{2}\rceil=\frac{n-1}{2}$ when n is odd which are the algebraic multiplicity of its eigenvalues. Then the characteristic polynomial $P(\mathscr{G},\lambda)$ will be

$$P(\mathscr{G},\lambda)=\begin{cases}(\lambda+1)^{\frac{n}{2}}(\lambda-1)^{\frac{n}{2}} & \text{if } n \text{ is even};\\ \lambda(\lambda+1)^{\frac{n-1}{2}}(\lambda-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.}\end{cases}$$

Whenever, we let $\det\left(\lambda\mathbb{I}-M(\mathscr{G})\right)=0$. Then, $\lambda=-1,1$ for even number of vertices with spectrum $(-1)^{\frac{n}{2}}$, $(1)^{\frac{n}{2}}$ and $\lambda=0,-1,1$ for odd number of vertices with spectrum $(-1)^{\frac{n-1}{2}},0,(1)^{\frac{n-1}{2}}$. This implies that $-1\le\lambda_i(\mathscr{G})\le 1$, where $\lambda_i(\mathscr{G})\in\mathbb{Z}$ is the eigenvalues of $M(\mathscr{G})$. □

Theorem 2.4. *Let $\mathscr{E}_M(\mathscr{G})$ be the mixed energy of a mixed hourglass graph $\mathscr{G}$ of order n. Then*

$$\mathscr{E}_M(\mathscr{G})=n-\gamma=\begin{cases}n & \textit{if } n \textit{ is even};\\ n-1 & \textit{if } n \textit{ is odd},\end{cases}$$

where $\gamma=\left|(n+1) \bmod 2-1\right|$.

Proof. Let $\lambda_i(\mathscr{G})=\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ be the mixed eigenvalues of a mixed hourglass-adjacency matrix. We know that $\lambda_i(\mathscr{G}) = -1, 1$ with $\frac{n}{2}$ multiple when n is even. Let $\frac{n}{2} = k$ then $\lambda_i(\mathscr{G})$ is -1 and 1 each in k times. Thus,

$$\begin{aligned}
\sum_{i=1}^{n} |\lambda_i(\mathscr{G})| &= \sum_{k=1}^{\frac{n}{2}} \left|(-1)^k + (1)^k\right| \\
&\le \left|\sum_{k=1}^{\frac{n}{2}} (-1)^k\right| + \left|\sum_{k=1}^{\frac{n}{2}} (1)^k\right| \\
&= \left|(-1)^k\right| \left|\sum_{k=1}^{\frac{n}{2}} (1)\right| + \left|(1)^k\right| \left|\sum_{k=1}^{\frac{n}{2}} (1)\right| \\
&= 2k \\
&= 2\left(\frac{n}{2}\right) \\
&= n.
\end{aligned}$$

Similarly, $\lambda_i(\mathscr{G}) = 0, -1, 1$ with $\frac{n-1}{2}$ multiple when n is odd. Let $\frac{n-1}{2} = p$ then $\lambda_i(\mathscr{G}) = 0, -1, 1$, where -1 and 1 appears each in p times. Thus,

$$\begin{aligned}
\sum_{i=1}^{n} \left|\lambda_i(\mathscr{G})\right| &= \sum_{p=1}^{\frac{n-1}{2}} \left|0 + (-1)^p + (1)^p\right| \\
&\le \left|\sum_{p=1}^{\frac{n-1}{2}} (-1)^p\right| + \left|\sum_{p=1}^{\frac{n-1}{2}} (1)^p\right| \\
&= \left|(-1)^p\right| \left|\sum_{p=1}^{\frac{n-1}{2}} (1)\right| + \left|(1)^p\right| \left|\sum_{p=1}^{\frac{n-1}{2}} (1)\right| \\
&= 2p \\
&= 2\left(\frac{n-1}{2}\right) \\
&= n-1.
\end{aligned}$$

Thus,

$$\mathscr{E}_M(\mathscr{G}) = \begin{cases} n & \text{if } n \text{ is even;} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Recall that

$$\gamma = \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Then,

$$n - \gamma = \begin{cases} n & \text{if } n \text{ is even;} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Therefore,

$$\mathscr{E}_M(\mathscr{G}) = n - \gamma \tag{2.4}$$

□

Corollary 2.5. *Let e and $\mathscr{E}_M(\mathscr{G})$ be the number of undirected edges and energy of a mixed hourglass graph respectively. Then*

$$\mathscr{E}_M(\mathscr{G}) = 2e.$$

Proof. From Lemma 2.1, $2e = n - \gamma$ and from Theorem 2.4, $\mathscr{E}_M(\mathscr{G}) = n - \gamma$. Where

$$\gamma = \left|(n+1) \bmod 2 - 1\right| = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

By suitable substitution of Equation (2.4) in Equation (2.3), we have

$$\mathscr{E}_M(\mathscr{G}) = 2e = \begin{cases} n & \text{if } n \text{ even;} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

□

Remark 2.1. If $M(G) = M(\mathscr{G})$, then $\sum\limits_{i=1}^{n} \lambda_i(G) = 0 = \sum\limits_{i=1}^{n} \lambda_i(\mathscr{G})$ and $\sum\limits_{i=1}^{n} \left(\lambda_i(G)\right)^2 = 2e = \sum\limits_{i=1}^{n} \left(\lambda_i(\mathscr{G})\right)^2$.

Corollary 2.6. *Let $\lambda_i(\mathscr{G})$ and $\mathscr{E}_M(\mathscr{G})$ be the eigenvalues of a mixed hourglass-adjacency matrix and mixed energy of a mixed hourglass graph $\mathscr{G}$ respectively. Then*

$$\mathscr{E}_M(\mathscr{G}) = \sum_{i=1}^{n} \left(\lambda_i(\mathscr{G})\right)^2.$$

Proof. From Corollary 2.5, $\mathscr{E}_M(\mathscr{G}) = 2e$ and we know from Remark 2.1 that $\sum\limits_{i=1}^{n} \left(\lambda_i(\mathscr{G})\right)^2 = 2e$. Thus,

$$\mathscr{E}_M(\mathscr{G}) = 2e = \sum_{i=1}^{n} \left(\lambda_i(\mathscr{G})\right)^2.$$

□

3. Conclusion

A mixed hourglass graph is a singular graph when the number of vertices is odd and a nonsingular graph when the number of vertices is even. Moreover, the energy of a mixed hourglass graph coincides with twice the number of edges in the graph and the sum of the square of its eigenvalues. Lastly, the energy of a mixed hourglass graph is also the sum of all entries in the anti-diagonal of a mixed hourglass-adjacency matrix. Laplacian matrix and Laplacian energy could be derived from mixed hourglass graph as well as the Zagreb index and k-factorization of the graph can be considered.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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