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Numerical Approximation of Fractional Burgers Equation

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Abstract. The mathematical modeling of physical and chemical systems is used extensively throughout science, engineering and applied mathematics. In order to make use of mathematical models, it is necessary to have solutions to the model equations. Generally, this requires numerical methods because of the complexity and number of equations. In this paper, we study and approximate a nonlinear fractional Burgers problem by finite volume schemes of order one in space and also in time. The purpose is to show that they converge to the solution of the considered problem and to establish error estimates. We prove that the finite volume schemes converge to weak entropic solutions as the discretization parameters tend to zero.

1. **Introduction**

The mathematical modeling of physical and chemical systems is used extensively throughout science, engineering and applied mathematics ([10], [11]). In order to make use of mathematical models, it is necessary to have solutions to the model equations. Generally, this requires numerical methods because of the complexity and number of equations.

In this paper, we study and approximate a nonlinear fractional Burgers problem by finite volume schemes of order one in space and also in time. The purpose is to show that they converge to the solution of the considered problem and to establish error estimates. In order to model solutions of Navier-Stokes equations, several authors ([1], [2], [3], [4], [5], [6], [7], [8] and [13]) have studied fractional Burgers equations with initial or random initial conditions.

Burgers equations involving in their linear parts fractional powers $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ of the Laplacian, $\alpha \in (0,2]$, have been investigated in connection with certain models of hydrodynamical phenomena (see [5] and its bibliography).

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Burgers equations in financial mathematics arise in connection with the behavior of the risk premium of the market portfolio of risky assets under Black-Scholes assumptions. These problems arise in a variety of engineering analysis and design situations.

We approach $(D^{\alpha}v)(x)$ on a mesh K by a quadrature formula. A new quadrature formula has been proposed which uses weight functions. For a time-discretization, we apply one of the various basic schemes to solve a general ordinary differential equation.

We prove that the finite volume schemes converge to weak entropic solutions as the discretization parameters tend to zero.

2. **Main Results**

2.1. *Numerical Approximation by Method of Lines (MOL) of Finite Volumes*

The basic idea of the MOL is to replace the spatial (boundary value) derivatives in the PDE with algebraic approximations. Once this is done, the spatial derivatives are no longer stated explicitly in terms of the spatial independent variables. Thus, in effect only the initial value variable, typically time in a physical problem, remains. In other words, with only one remaining independent variable, we have a system of ODEs that approximate the original PDE. The challenge, then, is to formulate the approximating system of ODEs. Once this is done, we can apply any integration algorithm for initial value ODEs to compute an approximate numerical solution to the PDE. Thus, one of the salient features of the MOL is the use of existing, and generally well established, numerical methods for ODEs.

Consider the following system:

$$\begin{cases} u_t = -D^{\alpha}u - \frac{1}{2}(u^2)_x + f(x,t), & 0 < \alpha \le 1/2 \text{ and} \\ & (x,t) \in [0,1] \times [0,T] \\ \begin{cases} u(0,t) = g_0(t), & t \in [0,T] \\ u(1,t) = g_1(t), & t \in [0,T] \\ u(x,0) = U_0(x), & x \in [0,1], \end{cases} \end{cases} \tag{2.1}$$

or under the following general hyperbolic form:

$$\begin{cases} u_t = -(G(u))_x + f(x,t), & 0 < \alpha \le 1/2 \text{ and} \\ & (x,t) \in [0,1] \times [0,T] \\ \begin{cases} u(0,t) = g_0(t), & t \in [0,T] \\ u(1,t) = g_1(t), & t \in [0,T] \\ u(x,0) = U_0(x), & x \in [0,1], \end{cases} \end{cases} \tag{2.2}$$

where

$$\left(\frac{u^2}{2}\right)_x + D^{\alpha}u = \frac{\partial}{\partial x}G(u) \quad \text{and} \quad D^{\alpha} \equiv (-\partial^2/\partial x^2)^{\frac{\alpha}{2}}$$

and D^α is defined by

$$(D^\alpha v)(x) = \frac{-1}{\Gamma(\alpha)} \int_0^x (x-z)^{\alpha-1} v(z)dz.$$

A weak solution $u(x,t)$ is defined on $[0,1] \times [0,T]$.

Consider a partition $(x_i)_{i\in\mathbb{N}}$ of $[0,1]$ and denote $x_{i+1/2} = \frac{x_{i+1}+x_i}{2}$.

Then, define the volume of control $K_i = [x_{i-1/2}, x_{i+1/2}]$ and its width $h_i = x_{i+1/2} - x_{i-1/2}$.

The union of volumes K_i is $[0,1]$. For every element K_i of mesh, we introduce a mean value $u_i(t)$ of a solution, that one assumes exist, in the following meaning:

$$u_i(t) = \frac{1}{h_i} \int_{K_i} u(x,t)dx, \qquad i \in \mathbb{N},$$

$$\frac{\partial u_i(t)}{\partial t} = \frac{1}{h_i} \int_{K_i} \frac{\partial u(x,t)}{\partial t} dx, \quad i \in \mathbb{N}.$$

Now we go to approach $(D^\alpha v)(x)$ on K_i (namely $x \in K_i$) by a quadrature formula. A new quadrature formula has been proposed which uses weight functions. This formula has the form given below:

$$(D^\alpha v)(x) = \frac{-1}{\Gamma(\alpha)} \int_0^x (x-z)^{\alpha-1} v(z)dz = \frac{-1}{\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i}(x-sh)^{\alpha-1} v(sh),$$

where sh are nodes of a quadrature formula and $w_{s,i}$ are weight functions with $\sum_{s=0}^{i} w_{s,i} = 1$.

Integrate equation (2.1) on K_i in order to obtain

$$\int_{K_i} u_t dx = -\int_{K_i} D^\alpha u dx - \int_{K_i} \frac{1}{2}(u^2)_x dx + \int_{K_i} f(x,t)dx$$

implies

$$\int_{K_i} u_t dx = \frac{1}{\Gamma(-\alpha)} \int_{K_i} \int_0^x (x-z)^{-\alpha-1} v(z)dzdx - \int_{K_i} \frac{1}{2}(u^2)_x dx + \int_{K_i} f(x,t)dx,$$

namely,

$$\int_{K_i} u_t dx = \frac{1}{\Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t) \int_{K_i} (x-x_s)^{\alpha-1} dx - \int_{K_i} \frac{1}{2}(u^2)_x dx + \int_{K_i} f(x,t)dx$$

implies

$$\begin{aligned} \int_{K_i} u_t dx \;=\; & \frac{1}{\alpha\Gamma(-\alpha)} \sum_{s=0}^{i} ((x_{i+1/2} - x_s)^\alpha - (x_{i-1/2} - x_s)^\alpha) w_{s,i} u(x_s,t) \\ & - \frac{1}{2}(u^2(x_{i+1/2},t) - u^2(x_{i-1/2},t)) + \int_{K_i} f(x,t)dx \end{aligned}$$

implies

$$\begin{aligned}\int_{K_i} u_t dx =& \left(\frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i+1/2}-x_s)^\alpha w_{s,i}u(x_s,t)-\frac{1}{2}u^2(x_{i+1/2},t)\right)\\ &-\left(\frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i-1/2}-x_s)^\alpha w_{s,i}u(x_s,t)+\frac{1}{2}(u^2(x_{i-1/2},t))\right)\\ &+\int_{K_i} f(x,t)dx\end{aligned}$$

Denote the flux of exact solution at the vertex $x_{i+1/2}$ and $x_{i-1/2}$ by

$$\begin{aligned}&F(x_{i+1/2},t)\\ &= \left(\frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i+1/2}-x_s)^\alpha w_{s,i}u(x_s,t)-\frac{1}{2}u^2(x_{i+1/2},t)\right)\\ &= \left(\frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i+1/2}-x_s)^\alpha w_{s,i}u(x_s,t)-\frac{1}{2}\left(\frac{u(x_{i+1},t)+u(x_i,t)}{2}\right)^2\right)\\ &= \left(\frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i+1/2}-x_s)^\alpha w_{s,i}u(x_s,t)-\frac{1}{8}(u(x_{i+1},t)+u(x_i,t))^2\right)\end{aligned}$$

and by

$$\begin{aligned}&F(x_{i-1/2},t)\\ &= \frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i-1/2}-x_s)^\alpha w_{s,i}u(x_s,t)+\frac{1}{2}(u^2(x_{i-1/2},t))\\ &= \frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i-1/2}-x_s)^\alpha w_{s,i}u(x_s,t)+\frac{1}{2}\left(\frac{u(x_i,t)+u(x_{i-1},t)}{2}\right)^2\\ &= \frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i-1/2}-x_s)^\alpha w_{s,i}u(x_s,t)+\frac{1}{8}(u(x_i,t)+u(x_{i-1},t))^2. \qquad (2.3)\end{aligned}$$

As x_i is the midpoint of K_i, one has

$$|u_i(t)-u(x_i,t)|\leq ch^2.$$

Approach the flux of an exact solution at the vertex $x_{i+1/2}$ by the numerical flux which depends on mean values $\Xi(u_0,u_1,\dots,u_i,u_{i+1})$

$$\begin{aligned}&\Xi(u_0,u_1,\dots,u_i,u_{i+1})\\ &= \left(\frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i+1/2}-x_s)^\alpha w_{s,i}u_s(t))-\frac{1}{8}(u_{i+1}(t)+u_i(t)\right)^2\\ &= \left(\frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}(x_{i+1/2}-x_s)^\alpha w_{s,i}u_s(t))-\Psi(u_{i+1}(t),u_i(t)\right)\end{aligned}$$

and at the vertex $x_{i-1/2}$ by the numerical flux which depends on mean values $\Xi(u_0, u_1, \ldots, u_{i-1}, u_i)$:

$$\begin{aligned}
&\Xi(u_0, u_1, \ldots, u_{i-1}, u_i) \\
&= \left(\frac{1}{\alpha\Gamma(-\alpha)} \sum_{s=0}^{i} (x_{i-1/2} - x_s)^\alpha w_{s,i} u_s(t)) - \frac{1}{8}(u_i(t) + u_{i-1}(t) \right)^2 \\
&= \left(\frac{1}{\alpha\Gamma(-\alpha)} \sum_{s=0}^{i} (x_{i-1/2} - x_s)^\alpha w_{s,i} u_s(t)) - \Psi(u_{i-1}(t), u_i(t) \right),
\end{aligned}$$

to obtain an ordinary differential equation

$$\frac{\partial u_i(t)}{\partial t} = \frac{1}{h_i}\Xi(u_0, u_1, \ldots, u_i, u_{i+1}) - \frac{1}{h_i}\Xi(u_0, u_1, \ldots, u_{i-1}, u_i) + f_i(t), \quad t \geq 0,$$

where

$$f_i(t) = \frac{1}{h_i}\int_{K_i} f(x,t)dx.$$

For a time-discretization, we apply one of the various basic schemes to solve a general ordinary differential equation:

$$\begin{cases} \dfrac{d\overrightarrow{X}}{dt} = F(t, \overrightarrow{X}) \\ \overrightarrow{X}(0) = \overrightarrow{X}_0. \end{cases}$$

2.2. *Stability for Explicit Schemes*

Theorem 2.1. *Let the assumptions* (H_1) *and* (H_2) *holds:*

$(H_1) : u^0 \in L^\infty([0,1])$;

(H_2) : *a condition C-F-L (Courant-Friedrichs-Lewy)*

$$\Delta t \leq \frac{4\inf_{i\in\mathbb{N}} h_i}{L_{m_1}}$$

where L_{m_1} *is a Lipschitz constant, takes place, then a solution* u_i^n *defined by*

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{h_i}\Xi(u_0^n, u_1^n, \ldots, u_i^n, u_{i+1}^n) - \frac{\Delta t}{h_i}\Xi(u_0^n, u_1^n, \ldots, u_i^n, u_i^n) + \Delta t f_i(t_n) \tag{2.4}$$

and

$$u_i^0 = \frac{1}{h_i}\int_{K_i} U_0(x)dx, \quad i \in \mathbb{N} \tag{2.5}$$

verifies

$$A \leq u_i^n \leq B, \quad \text{for all } n,\ i \in \mathbb{N},$$

and

$$\|u_i^n\|_\infty \leq 2^{N_{\max}}\|U_0\|_\infty + \left(\sum_{j=0}^{N_{\max}} 2^j\right)\|f_i(t_n)\|_\infty \leq B.$$

Proof. According to assumption $A \leq U_0 \leq B$, a.e. and a definition of u_i^0, we see that $\overline{A} \leq u_i^0 \leq \overline{B}$, for all $i \in \mathbb{N}$. Let $n \in \mathbb{N}$. Suppose

$$\|u_i^n\|_\infty \leq \|U_0\|_\infty + n_{\max}\|f_i(t_n)\|_\infty, \quad \text{for all } \ i \in \mathbb{N}.$$

Let us show that this property is still true in the rank $n+1$. We have

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{h_i}\Xi(u_0^n, u_1^n, \dots, u_i^n, u_{i+1}^n) - \frac{\Delta t}{h_i}\Xi(u_0^n, u_1^n, \dots, u_{i-1}^n, u_i^n) + \Delta t f_i(t_n)$$

and

$$\begin{aligned} u_i^{n+1} &= u_i^n + \Delta t f_i(t_n) + \frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i} w_{s,i}((x_{i+1/2} - x_s)^\alpha - (x_{i-1/2} - x_s)^\alpha)u_s^n \\ &+ \frac{\Delta t}{8h_i}((u_i^n + u_{i-1}^n)^2 - (u_i^n + u_i^n)^2 + (u_i^n + u_i^n)^2 - (u_{i+1}^n + u_i^n)^2) \end{aligned}$$

Let us explicit $((x_{i+1/2} - x_s)^\alpha - (x_{i-1/2} - x_s)^\alpha)$. Then, we have

$$\begin{aligned} &(x_{i+1/2} - x_s)^\alpha - (x_{i-1/2} - x_s)^\alpha \\ &= \frac{(x_{i+1/2} - x_s)^\alpha - (x_{i-1/2} - x_s)^\alpha}{(x_{i+1/2} - x_{i-1/2})}(x_{i+1/2} - x_{i-1/2}) \\ &= \alpha(x_{i+1/2} - x_{i-1/2})^{\alpha-1}(x_{i+1/2} - x_{i-1/2}) = \alpha {h_i}^\alpha. \end{aligned}$$

Therefore

$$\begin{aligned} u_i^{n+1} &= u_i^n + \Delta t f_i(t_n) + \frac{1}{\Gamma(-\alpha)}\sum_{s=0}^{i} w_{s,i}{h_i}^\alpha u_s^n + \frac{\Delta t}{8h_i}((u_i^n + u_{i-1}^n)^2 \\ &-(u_i^n + u_i^n)^2 + (u_i^n + u_i^n)^2 - (u_{i+1}^n + u_i^n)^2), \end{aligned}$$

hence

$$\begin{aligned} u_i^{n+1} &= (1 - b_{i+\frac{1}{2}} - a_{i-\frac{1}{2}})u_i^n + b_{i+\frac{1}{2}}u_{i+1}^n(t) + a_{i-\frac{1}{2}}u_{i-1}^n(t) \\ &+ \sum_{s=0}^{i}\beta(\alpha, i, s)u_s^n + \Delta t f_i(t_n), \end{aligned}$$

where

$$b_{i+\frac{1}{2}} = \begin{cases} \dfrac{\Delta t}{8h_i}\left(\dfrac{(u_{i+1}^n + u_i^n)^2 - (u_i^n + u_i^n)^2}{u_i^n - u_{i+1}^n}\right), & \text{if } \ u_{i+1}^n \neq u_i^n, \\ 0, & \text{if } \ u_{i+1}^n = u_i^n; \end{cases}$$

and

$$a_{i-\dfrac{1}{2}} = \begin{cases} \dfrac{\Delta t}{8h_i}\left(\dfrac{(u_{i-1}^n + u_i^n)^2 - (u_i^n + u_i^n)^2}{u_{i-1}^n - u_i^n}\right), & \text{if } \ u_{i-1}^n \neq u_i^n, \\ 0, & \text{if } \ u_{i-1}^n = u_i^n, \end{cases}$$

and

$$\beta(\alpha,i,s)=\frac{1}{\Gamma(-\alpha)}w_{s,i}{h_i}^{\alpha}.$$

We have

$$b_{i+\frac{1}{2}}\leq L_{m_1}\frac{\Delta t}{8h_i},\quad a_{i-\frac{1}{2}}\leq L_{m_1}\frac{\Delta t}{8h_i},\quad \text{for all}\;\; i\in\mathbb{N}$$

and

$$\sum_{s=0}^{N_{\max}}\beta(\alpha,i,s)\leq 1.$$

$\Psi(p,q)$ is a Lipschitz function on $[A,B]^2$ with the same Lipschitz constant in p and q: L_{m_1}. Consequently, u_i^{n+1} is a convex combination of u_i^n, u_{i+1}^n and u_{i-1}^n on one part and linear combination on the other part. Then, a recurrence assumption implies

$$\|u_i^{n+1}\|_\infty\leq 2\|u_i^n\|_\infty+\|f_i(t_n)\|_\infty$$

and gives us

$$\begin{aligned}\|u_i^n\|_\infty &\leq\; 2^{N_{\max}}\|U_0\|_\infty+\sum_{j=0}^{N_{\max}}2^j\|f_i(t_n)\|_\infty\\ &\leq\; 2^{N_{\max}}\|U_0\|_\infty+n_{\max}\|f_i(t_n)\|_\infty\leq B.\qquad\square\end{aligned}$$

2.3. *Convergence*

Fix an initial condition $u^0\in L^\infty([0,1])$, that we discretize on a mesh $\mathscr{T}_m$ of step $h_m>0$:

$$(u_m)_i^0=\frac{1}{h_i}\int_{K_i}u(x,0)dx,\quad i\in\mathbb{N}.$$

We use a step of time Δt_m and search a function u_m supposed constant on every product of form $]ih_m,(i+1)h_m[\,\times\,]n\Delta t_m,(n+1)\Delta t_m[$:

$$u_m(x,t)=(u_m)_i^n,\quad (x,t)\in]ih_m,(i+1)h_m[\,\times\,]n\Delta t_m,(n+1)\Delta t_m[.$$

Calculate $(u_m)_i^{n+1}$

$$\frac{1}{\Delta t_m}((u_m)_i^{n+1}-(u_m)_i^n)=\frac{1}{h_m}\Xi(u_0^n,u_1^n,\dots,u_i^n,u_{i+1}^n)-\frac{1}{h_m}\Xi(u_0^n,u_1^n,\dots,u_{i-1}^n,u_i^n)+f_i^{\,n}.$$

When $\Delta t_m\to 0$ and $h_m\to 0$ for $m\to\infty$, the functions family $(u_m)_{m\in\mathbb{N}}$ can converge to a weak solution of problem.

Theorem 2.2 (Convergence). *Let us situated in the frame above of a family $\mathcal{T}_m$ of uniform mesh of space-step $h_m > 0$ and a time-step Δt_m. Construct a sequence (u_m). Let the assumptions (h_1), (h_2), (h_3) and (h_4) holds:*

(h_1) : *numerical flux function $\Xi(\cdot,\cdot)$ is consistent with the flux $(\frac{u^2}{2} + \int D^\alpha u)$, of conservation law:*

$$\Xi(u,\dots,u) = \left(\frac{u^2}{2} + \int D^\alpha u\right) = G(u);$$

(h_2) : *a sequence $(u_m)_{m\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,1])$*

$$\exists\, K > 0,\ \ \|u^0\|_{L^\infty([0,1])} \le d, \quad \|u_m\|_{L^\infty([0,1])} \le d, \ \ m \in \mathbb{N};$$

(h_3) : *a numerical flux function Ψ is Lipschitz on $[-d,d]^2$:*

$$\exists\, L > 0,\ \ |\Xi(u_1,u_2,\dots,u_n) - \Xi(v_1,v_2,\dots,v_n)| \le L\left(\sum_{p=1}^{n}|u_p - v_p|\right),$$

$$(u_1,u_2,\dots u_n; v_1,v_2,\dots,v_n) \in [-d,d]^{2n};$$

(h_4) : *it exists $u \in L^\infty([0,1]\times[0,T])$ such that*

$$\lim_{(m\to\infty)} u_m(x,t) = u(x,t), \quad a.e. \ \ (x,t)$$

Then $u(\cdot,\cdot)$ is a weak solution to problem.

Remark 2.3. The previous theorem expresses that if the family of numerical solutions $(u_m)_{m\in\mathbb{N}}$ converges, it is towards a weak solution of the problem.

Definition 2.4 (Consistency with the Condition of Entropy). An explicit scheme of finite volume is consistent with entropy inequality if, for any mathematical entropy η, there is a function of numerical flux of entropy Φ consistent with the flux of entropy ξ satisfying

$$(\xi' = \eta'(G(u))') : \Phi(u,\dots,u) = \xi(u)$$

such that if u_i^n is given by the following numerical scheme:

$$\frac{1}{\Delta t}(u_i^{n+1} - u_i^n) = \frac{1}{h_i}\Xi(u_0^n, u_1^n, \dots, u_i^n, u_{i+1}^n) - \frac{1}{h_i}\Xi(u_0^n, u_1^n, \dots, u_{i-1}^n, u_i^n) + f_i^n$$

then we have, also, a discrete inequality of entropy

$$\frac{1}{\Delta t}(\eta(u_i^{n+1}) - \eta(u_i^n)) + \frac{1}{h_i}(\Phi(u_{i+1}^n, u_i^n) - \Phi(u_i^n, u_{i-1}^n)) \le \eta(f_n^i),$$

where

$$\eta(f_n^i) = \frac{1}{h_i \Delta t}\int_{t_n}^{t_{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \eta'(u) f(x,t)\,dx\,dt.$$

This definition allows to complete the Lax-Wendroff's theorem ([12]) on uniqueness of solution in a fractional case.

Proposition 2.5 (Convergence of Finite Volume Methods). *Under assumptions of Theorem 2.2, if the numerical scheme is consistent with the condition of entropy, for any entropy η, the numerical flux Φ is Lipschitz on an interval $[-d,d]$, then the limit u is a bounded unique weak entropic solution of a problem.*

Proposition 2.6 (Sufficient Condition of Entropic Consistency for the Method of Lines). *Let $U \in \Omega \subset \mathbb{R}^p \to \eta(U) \in \mathbb{R}$ be a mathematical entropy. Denote $U \in \Omega \to \pi(U) = \xi = (\xi_1, \dots, \xi_p) \in \mathbb{R}^p$ the vector of entropic variables,*

$$\pi_i(U) = \xi_i = \frac{\partial \eta(U)}{\partial u_i}.$$

If the function of numerical flux satisfies

$$D_{k_{i-1},k_{i+1}} \equiv \int_{\xi_{k_{i-1}}}^{\xi_{k_{i+1}}} ((\Xi(\xi_0^n, \xi_1^n, \dots, \xi_i^n, \xi_{i+1}^n) - \Xi(\xi_0^n, \xi_1^n, \dots, \xi_{i-1}^n, \xi_i^n)) - \langle \xi, \nabla \xi \rangle) d\xi \leq 0,$$

for every couple $(\xi_{i+1}^n, \xi_{i-1}^n)$ of entropic variables, then the method of lines is consistent with the condition of entropy.

Remark 2.7. E. Tadmor ([14]) showed that any scheme of implicit finite volume is consistent with the inequality of entropy.

2.4. *Explicit and Implicit Error Estimates*

In this section, we prove estimates of error in $L^1_{loc}([0,1] \times [0,T])$ between the approached solution obtained by an explicit or implicit scheme and an entropic solution.

Theorem 2.8 (Error Estimate for An Explicit Scheme). *Let u be a solution of* (2.1) *satisfying*

$$\begin{aligned}
&\int_0^T \int_0^1 \left(|u-k| \frac{\partial \varphi}{\partial t} + sg_0(u-k) \left((u^2 - k^2) \frac{\partial \varphi}{\partial x} + f(x,t)\varphi \right) \right) dx dt \\
&+ \int_0^1 |u_0(x) - k| \varphi(x,0) dx \\
&\qquad \geq \int_0^T sg_0(u_2(t) - k) \left(\frac{u_2^2(t) - \chi_{u_1}(t)^2}{2} \right) \varphi(1,t) dt \\
&\qquad\qquad - \int_0^T sg_0(u_1 - k) \left(\frac{u_1^2(t) - \chi_{u_0}(t)^2}{2} \right) \varphi(0,t) dt;
\end{aligned}$$

$\chi_{u_r}(t)$ is the trace of $u(.,t)$ at r. Let $u_{\mathcal{T}} = u_i^n$, for $x \in K_i$ and $t \in]n\Delta t, (n+1)\Delta t[$, be an approached solution defined by the following explicit scheme

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{h_i} \Xi(u_0^n, u_1^n, \dots, u_i^n, u_{i+1}^n) - \frac{\Delta t}{h_i} \Xi(u_0^n, u_1^n, \dots, u_{i-1}^n, u_i^n) + \Delta t . f_i^n.$$

If $U_0 \in BV_{loc}(]0,1[) \cap L^\infty(]0,1[)$, we have the following error estimate: for any compact E of $(]0,1[\times]0,T[)$ there exists k_c which depends of E, U_0 and L_{m_1} such that

$$\int^E |u_{\mathcal{T}}(x,t) - u(x,t)| dx dt \leq k_c h^{\frac{1}{4}}.$$

Proof. A proof rests on the integral of continuous entropy satisfied by $u_{\mathcal{T}}(x,t)$ and on the following lemma:

Lemma 2.9. *Suppose $U_0 \in BV_{loc}(]0,1[) \cap L^\infty(]0,1[)$. Let $\tilde{u} \in L^\infty(]0,1[\times]0,T[)$ be such that $A \leq u_i^n \leq B$ almost everywhere. Suppose there exists $\sigma \in M(]0,1[\times]0,T[)$ and $\sigma_0 \in M(]0,1[)$ where $M(\Omega)$ is the set of positive continuous linear functionals on $C_c(\Omega)$ such that*

$$\begin{aligned}
&\int_0^T \int_0^1 \left(|u-k| \frac{\partial \varphi}{\partial t} + sg_0(u-k) \left(\left(\frac{u^2 - k^2}{2} \right) \frac{\partial \varphi}{\partial x} + f(x,t)\varphi \right) \right) dx dt \\
&+ \int_0^1 |u_0(x) - k| \varphi(x,0) dx \\
&\qquad \geq \int_0^T sg_0(u_2(t) - k) \left(\frac{g_1^2(t) - \chi_{u_1}(t)^2}{2} \right) \varphi(1,t) dt \\
&\qquad\quad - \int_0^T sg_0(u_1 - k) \left(\frac{g_0^2(t) - \chi_{u_0}(t)^2}{2} \right) \varphi(0,t) dt \\
&\qquad\quad - \int_0^T \int_0^1 \left(\left| \frac{\partial \varphi(x,t)}{\partial t} \right| + \left| \frac{\partial \varphi(x;t)}{\partial x} \right| \right) d\sigma(x,t) - \int_0^1 |\varphi(x,0)| d\sigma_0(x), \\
&\qquad\qquad\qquad \text{for all } \ k \in \mathbb{R}, \text{for all } \ \varphi \in C_c^\infty(]0,1[\times]0,T[).
\end{aligned}$$

Let $u \in L^\infty(]0,1[\times]0,T[)$ such that:

$$\begin{aligned}
&\int_0^T \int_0^1 \left(|u-k| \frac{\partial \varphi}{\partial t} + sg_0(u-k) \left(\left(\frac{u^2 - k^2}{2} \right) \frac{\partial \varphi}{\partial x} + f(x,t)\varphi \right) \right) dx dt \\
&+ \int_0^1 |u_0(x) - k| \varphi(x,0) dx \\
&\qquad \geq \int_0^T sg_0(u_2(t) - k) \left(\frac{g_1^2(t) - \chi_{u_1}(t)^2}{2} \right) \varphi(1,t) dt \\
&\qquad\quad - \int_0^T sg_0(u_1 - k) \left(\frac{g_0^2(t) - \chi_{u_0}(t)^2}{2} \right) \varphi(0,t) dt, \\
&\qquad\qquad\qquad \text{for all } \ k \in \mathbb{R}, \text{for all } \ \varphi \in C_c^\infty(]0,1[\times]0,T[).
\end{aligned}$$

Then, for any compact $E \subset]0,1[\times]0,T[$, there exist c_{E,u_0}, u, $\widetilde{r}$ and $\widetilde{T}$ independent of E, u and u_0 such that

$$\int^{E} |\widetilde{u}(x,t)-u(x,t)|dxdt$$
$$\leq c_{E,u_0}(\sigma_0(B(0,\widetilde{r}))+\sigma(B(0,\widetilde{r})\times[0,\widetilde{T}])+\sigma(B(0,\widetilde{r})\times[0,\widetilde{T}])^{\frac{1}{2}}).$$

Indeed; the following theorem proves the existence of measures $\sigma \in M(]0,1[\times]0,T[)$ and $\sigma_0 \in M(]0,1[)$:

Theorem 2.10. *Under previous assumptions on data and mesh, there exists measures $\mu_\tau \in M(]0,1[)$ and $\mu_{\tau,k} \in M(]0,1[\times]0,T[)$ such that for all $k \in \mathbb{R}$, for all $\varphi \in C_c^\infty(]0,1[\times]0,T[)$:*

$$\int_0^T\int_0^1 \left(|u_{\mathscr{T}}-k|\frac{\partial\varphi}{\partial t}+sg_0(u_{\mathscr{T}}-k)\left(\left(\frac{u_{\mathscr{T}}^2-k^2}{2}\right)\frac{\partial\varphi}{\partial x}+f(x,t)\varphi\right)\right)dxdt$$
$$+\int_0^1 |u_0(x)-k|\varphi(x,0)dx$$
$$\geq \int_0^T sg_0(g_1(t)-k)\left(\frac{g_1^2(t)-\chi_{u_1}(t)^2}{2}\right)\varphi(1,t)dt$$
$$-\int_0^T sg_0(g_0-k)\left(\frac{g_0^2(t)-\chi_{u_0}(t)^2}{2}\right)\varphi(0,t)dt-\int_0^T\int_0^1\left(\left|\frac{\partial\varphi(x,t)}{\partial t}\right|\right.$$
$$\left.+\left|\frac{\partial\varphi(x;t)}{\partial x}\right|\right)d\mu_{\tau,k}(x,t)-\int_0^1|\varphi(x,0)|d\mu_\tau(x)$$

Moreover,

(i) *for any ball B_r of radius, $\widetilde{r}>0$, $\widetilde{T}>0$; there exists c_m which depends of U_0, $\widetilde{r}$, L_m and $\widetilde{T}$ such that*

$$\mu_{\tau,k}(B_r\times[0,\widetilde{T}])\leq c_m(h+\sqrt{h}),\quad \text{for all } h<\widetilde{r};$$

(ii) *μ_τ is the measure of density $|u_{\mathscr{T},0}-U_0|$ (where $u_{\mathscr{T},0}=u_i^0$, for all $i\in\mathbb{N}$) relatively to Lebesgue measure. For any $\widetilde{r}>0$, one has then*

$$\lim_{(h\to 0)}\mu_\tau(B_r)=0.$$

If $U_0 \in BV_{loc}(]0,1[)\cap L^\infty(]0,1[)$, there exists D_m which depends of U_0 and $\widetilde{r}$ such that

$$\mu_\tau(B_r)\leq D_m h,\quad \text{for all } h<\widetilde{r}.$$

Error estimates verified by implicit scheme shall be obtained in the same way as explicit scheme. We have the following result:

Theorem 2.11 (Error Estimate for An Implicit Scheme)**.** *Let $u_{\mathscr{T}}=u_i^n$, while $x\in K_i$ and $t\in]n\Delta t,(n+1)\Delta t[$, be an approached solution defined by the following explicit* *scheme:*

$$\begin{aligned} u_i^{n+1} &= u_i^n + \Delta t \cdot f_i^{n+1} + \frac{\Delta t}{h_i}\Xi(u_0^{n+1}, u_1^{n+1}, \ldots, u_i^{n+1}, u_{i+1}^{n+1}) \\ &\quad - \frac{\Delta t}{h_i}\Xi(u_0^{n+1}, u_1^{n+1}, \ldots, u_{i-1}^{n+1}, u_i^{n+1}). \end{aligned}$$

If $U_0 \in BV_{loc}(]0,1[) \cap L^\infty(]0,1[)$, then we have the following error estimate: for any compact E of $(]0,1[\times]0,T[)$, there exists k which depends of E, U_0 and L_{m_1} such that

$$\int^E |u_{\mathscr{T}}(x,t) - u(x,t)|dxdt \leq k_c\left(\frac{\Delta t}{h} + \sqrt{h}\right)^{\frac{1}{2}}.$$

Remark 2.12. Tadmor ([14]) proved that any implicit finite volume scheme is consistent with inequality of entropy.

Proposition 2.13 (Convergence of finite volumes method)**.** *Under assumptions of previous theorem, if the numerical scheme is consistent with the entropic condition, for any entropy η where the numerical flux Φ is Lipschitz on the interval $[-d,d]$, then a limit u is a bounded unique and weak entropic solution of problem.*

2.5. *Numerical Results*

We give, below, some results of numerical simulations and error estimates for $\alpha = \frac{1}{2}$ and

$$f(x,t) = \left(\frac{2x^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{2\sqrt{x}}{\sqrt{\pi}} + x(x-1)(-\beta + (2x-1)\exp(-\beta * t))\right)\exp(-\beta * t).$$

We have the following numerical results:

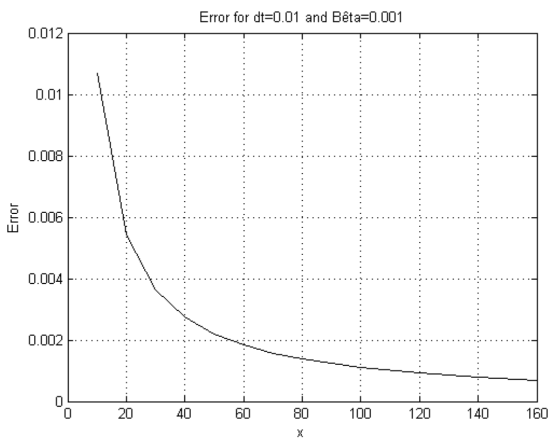

Figure 1

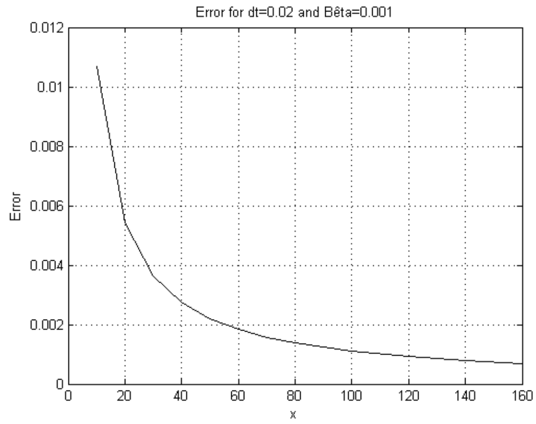

Figure 2

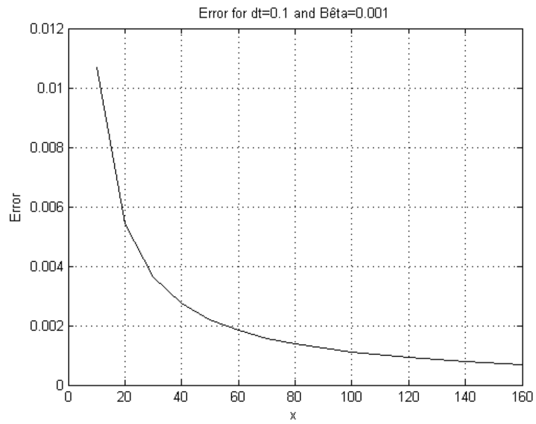

Figure 3

2.6. *Conclusions and Perspective*

We studied and give finite volumes approach on a regular domain then estimate errors made for the fractional Burgers equation.

In view of the importance of fractional Burgers equation in sciences of engineering and his various applications, we hope to develop these results, always by using the finite volumes technique, for perturbed fractional Burgers equation on a non regular domain.

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