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Power Series Method for Linear Partial Differential Equations of Fractional Order

Muhammet Kurulay and Mustafa Bayram

Abstract. In this article, a novel numerical method is proposed for linear partial differential equations with time-fractional derivatives. This method is based on power series and generalized Taylor's formula. The fractional derivatives are considered in the Caputo sense. Several illustrative examples are given to demonstrate the effectiveness of the present method. The modified algorithm provides approximate solutions in the form of convergent series with easily computable components. The obtained results are in good agreement with the existing ones in open literature and it is shown that the technique introduced here is robust, efficient and easy to implement.

1. **Introduction**

In the last several decades, many researchers have found that derivatives of noninteger order are very suitable for the description of various physical phenomena such as rheology, damping laws and diffusion process. These findings have invoked a growing interest of studies of the fractal calculus in some various fields such as physics, fluid mechanics, biology, chemistry, acoustics, control theory, chemistry and engineering. Several excellent books and papers describing the state-of-the-art available in the literature testify to the maturity of theory of fractal order. Podlubny [1, 13] provided the solution methods of differential equations of arbitrary real order and applications of the described methods in various fields. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations [2].

Several analytical and numerical methods have been proposed to solve fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. The most commonly used ones are; Adomian Decomposition Method (ADM) [4], Variational Iteration Method (VIM) [5], Fractional Difference Method (FDM) [1], Differential Transform Method (DTM) [6, 7], Homotopy Perturbation Method (HPM) [8]. Also there are some

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other classical solution techniques. These are Laplace transform method, Fractional Green's function method, Mellin transform method and method of orthogonal polynomials [1]. Among these solution techniques, the variational iteration method and the Adomian decomposition method are the most clear methods of solution of fractional differential and integral equations, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization. Recently some numerical methods have been developed to solve linear partial differential equations of fractional order and nonlinear partial differential equations of fractional order [9, 10, 12]. Homotopy analysis method is applied to solve fractional partial differential equations [11].

2. Fractional Calculus

There are several definitions of a fractional derivative of order [1, 3] e.g. Riemann-Liouville, Grunwald-Letnikow, Caputo and Generalized Functions Approach. The most commonly used definitions are the Riemann-Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number ($p > \mu$), such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it said to be in the space C_μ^m iff $f^m \in C_\mu, m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J_0^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \quad \nu > 0,$$
$$J^0 f(x) = f(x).$$

It has the following properties: For $f \in C_\mu, \mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > 1$:

(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
(2) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
(3) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for the physical problems of the real world since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition 2.3. The fractional derivative of $f(x)$ in the Caput sense is defined as

$$D_*^\nu f(x) = J_a^{m-\nu} D^m f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} f^{(m)}(t)dt,$$

for $m - 1 < \nu < m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$.

Lemma 2.4. *If* $m-1<\alpha<m$, $m\in N$ *and* $f\in C_\mu^m, \mu\geq -1$, *then*

$$D_*^\alpha J^\alpha f(x)=f(x),$$
$$J^\alpha D_*^\nu f(x)=f(x)-\sum_{k=0}^{m-1} f^k(0^+)\frac{x^k}{k!},\quad x>0$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we have considered the time-fractional linear partial differential equation, where the unknown function $u=u(x,t)$ is a assumed to be a causal function of time and the fractional derivatives are taken in Caputo sense as follows:

Definition 2.5. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha>0$ is defined as

$$D_{*t}^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\xi)^{m-\alpha-1}\frac{\partial^m u(x,\xi)}{\partial \xi^m}d\xi, & \text{for } m-1<\alpha<m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha=m\in\mathbb{N}. \end{cases}$$

3. **Power Series Method**

In this paper, we consider the solution of linear time-fractional partial differential equations of form [12]

$$\frac{\partial^\alpha u}{\partial t^\alpha}+a_0(x)u+a_1(x)\frac{\partial u}{\partial x}+a_2(x)\frac{\partial^2 u}{\partial x^2}+a_3(x)\frac{\partial^3 u}{\partial x^3}+\ldots+a_n(x)\frac{\partial^n u}{\partial x^n}=q(x,t),\quad t>0,\ x\in\mathbb{R} \tag{3.1}$$

subject to the initial and boundary conditions

$$u(x,0)=f(x),\ \ 0<\alpha\leq 1,\quad u(x,t)\to 0\ \text{ as } |x|\to\infty,\ t>0$$
$$u(x,0)=f(x),\ \frac{\partial u(x,0)}{\partial t}=g(x),\ 0<\alpha\leq 2,\quad u(x,t)\to 0\ \text{ as } |x|\to\infty,\ t>0$$

where $a_i(i=0,1,\ldots,n)$, $f(x)$, $g(x)$ and (x,t) all are continuous functions and α is a parameter describing the order of the time-fractional derivative. When $0<\alpha\leq 1$, equation(3.1) can be reduced to a fractional heat-like equation, and to a wave-like equation for $0<\alpha\leq 2$. In case of α=1, the fractional equation reduces to the classical linear partial differential equation.

The coefficients of variables of a function $u(x,y)$ are defined as follows:

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D_{*x_0}^\alpha)^k(D_{*y_0}^\beta)^h u(x,y)]_{(x_0,y_0)},$$

where $(D_{x_0}^\alpha)^k=D_{x_0}^\alpha D_{x_0}^\alpha\cdots D_{x_0}^\alpha$, k-times.

The function $u(x,y)$ can be represented as

$$u(x,y) = \sum_{k=0}^{\infty} F_\alpha(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_\beta(h)(y-y_0)^{h\beta} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta},$$

where $0<\alpha,\ \beta\le 1$, $U_{\alpha,\beta}(k,h)=F_\alpha(k)G_\beta(h)$ is called the spectrum of $u(x,y)$.

From equations (3.2) and (3.2), we have

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D^{\alpha}_{*x_0})^k(D^{\beta}_{*y_0})^h u(x,y)]_{(x_0,y_0)} \times (x-x_0)^{k\alpha}(y-y_0)^{h\beta},$$

where it is to noted that upper case symbol $U(k,h)$ is used to denote the coefficients of variables in (3.2) which are represented by a corresponding lower case symbol original function $u(x,y)$.

This section aims at describing a numerical solution of time-fractional derivative partial differential equations by power series method. We write power series method in the form

$$u(x,t) = U_0+U_1x+U_{01}t^\alpha+U_{11}xt^\alpha+\cdots+U_{mn-1}x^m t^{(n-1)\alpha}+ax^m t^{n\alpha}, \quad k-1<\alpha<k,\ m,\ n,\ k\in\mathbb{N} \quad (3.2)$$

where $U_0, U_1, U_{01}, U_{11}, \ldots$ are known constants but a is an unknown constant. Substituting into (1), we can get the following:

$$U(m,n)=(\mu a+\lambda)x^m t^{n\alpha-i}=0$$

where μ and λ are constant and i is the order of the partial differential equation. From (8), we have a constant. Substituting (3.3) into (3.2), we get the solution of the partial differential equation. Repeating this procedure from (3.2)-(3.3), we can get the power series method of the solution for time-fractional derivative PDEs in (3.1).

4. A Numerical Solution of Second-order Partial Differential Equations

4.1. *Example*

Consider the linear time-fractional equation [11]

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}, \quad t>0,\ 0<x<1,\ 0<\alpha\le 1,$$

Subject to the initial condition

$$u(x,0)=x^2,$$

and the boundary conditions

$$u(0,t)=0, \quad u(1,t)=e^t.$$

Using equation (3.2), and initial condition , we obtain

$$U(i,0)=0, \qquad i=0,1,3,4,\dots,m, \tag{4.1}$$
$$U(2,0)=1.$$

Substituting equations (4.4) into equations (3.2), we have

$$u_1(x,t)=x^2+ax^2t^\alpha.$$

Substituting equations(4.2) into equations (4.1),

$$U(2,1)=\frac{1}{\Gamma(\alpha+1)}$$

and by recursive method, the results corresponding to $n\to\infty$ are listed as follows:

$$\begin{aligned} u(x,t) &= x^2\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots\right] \\ &= x^2\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \end{aligned}$$

4.2. *Example*

We next consider linear inhomogeneous time-fractional equation [12]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+x\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}=2t^\alpha+2x^2+2, \quad t>0,\ 0<\alpha\le 1,$$

subject to the initial condition

$$u(x,0)=x^2.$$

Using equation (3.2), and initial condition, we obtain

$$U(i,0)=0, \quad i=0,1,3,4,\dots,m, \tag{4.2}$$
$$U(2,0)=1.$$

Substituting equations (4.9) into equations (3.2), we have

$$u_1(x,t)=x^2+at^{2\alpha}.$$

Substituting equations(4.3) into equations (4.2),

$$a=2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}$$

and we get the results corresponding to $m\to\infty$, $n\to\infty$ are listed as follows:

$$u(x,t)=x^2+2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}.$$

Which is the exact solution of the linear-inhomogeneous time-fractional equation (4.2).

5. **Conclusions**

A new generalization of the power series method has been developed for linear partial differential equations with time-fractional derivatives. The new method is based on the power series method, generalized Taylor's formula and Caputo fractional derivative. It may be concluded that the power series is very powerful efficient technique in finding exact and approximate solutions for ordinary and partial differential equations of fractional order. Although the method is well suited to solve the time-fractional equation in terms of a rapid convergent series with easily computable components, the method could lead to a promising approach for many applications in applied sciences.

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Muhammet Kurulay, *Yildiz Technical University, Faculty of Art and Sciences Department of Mathematics, 34210-Davutpasa- stanbul, Turkey*.
E-mail: mkurulay@yildiz.edu.tr

Mustafa Bayram, *Fatih University, Faculty of Arts and Science, Department of Mathematics, 34500-B y k ekmece-Istanbul, Turkey*.
E-mail: mbayram@fatih.edu.tr

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