Communications in Mathematics and Applications
Vol. 10, No. 4, pp. 681–691, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i4.1129

Research Article

Numerical Solution of Singularly Perturbed Differential-Difference Equations using Multiple Fitting Factors

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Abstract. In this paper, a numerical scheme is proposed to solve singularly perturbed differential-difference equations with boundary layer behaviour using two fitting factor inserted at convective and diffusion terms. The singularly perturbed differential difference equation is replaced by an equivalent two point singularly perturbation problem. Then to handle the boundary layer, a two parameter fitted scheme is derived and it is applied to get the accurate solution. Model examples are solved using this approach and numerical results along with graphical representation are shown to support the method.

Keywords. Singularly perturbed differential-difference equations; Boundary Layer; Fitting factor

MSC. 65L10; 65L11; 65L12

Received: November 16, 2018 **Accepted:** April 3, 2019

1. Introduction

A singularly perturbed differential-difference equation is a differential equation in which the highest order derivative terms is multiplied by a positive small parameter and involving at least one delay or advance term or both. Such problems arise frequently in the study of human pupil light reflex [9], control theory [2], mathematical biology [11], etc. The mathematical modeling of the determination of the expected time for the generation of action potentials in nerve cells by random synaptic inputs in dendrites includes a general boundary value problem for singularly perturbed differential-difference equation with small shifts. Different numerical methods were proposed to solve singularly perturbed problems by Roberts [18], Bender and Orszag [1], O'Malley [12], and Miller et al. [10].

In [8], Lange and Miura considered boundary-value problems for singularly perturbed linear second-order differential-difference equations with small shifts. In particular, this paper describes on problems with solutions that exhibit layer behavior at one or both of the end points of the interval. The analyses of the layer equations using Laplace transform lead to novel results.

Numerical study for approximating the solution of SPDDE given by Kadalbajoo and Sharma [6] with mixed shifts. In [4] Kadalbajoo and Kumar presented a technique based on piecewise uniform mesh and quasilinearization process for SPDDE with small shifts. Pramod et al. [13] presented an exponentially fitted finite difference scheme to solve second order singularly perturbed delay differential equation with large delay using cubic spline in compression. Ravi Kanth and Murali [17] presented a numerical scheme for a singularly perturbed convection delayed dominated diffusion equation via tension splines. In this method, a fitting factor is introduced to the highest derivative of the differential equation and solved by tension splines.

Chakravarthy and Rao [14] proposed a modified fourth order Numerov method is presented for solving singularly perturbed differential-difference equations of mixed type. Authors constructed a special type of mesh, so that the terms containing shift lie on nodal points after discretization. This finite difference method works nicely when the delay parameter is smaller or bigger to perturbation parameter. In [16], Ravi Kanth and Murali has given a numerical method based on parametric cubic spline for a class of nonlinear singularly perturbed delay differential equations. Quasilinearization process is applied to reduce the nonlinear singularly perturbed delay differential equations into a sequence of linear singularly perturbed delay differential equations. To handle the delay term, they have constructed a special type of mesh in such a way that the term containing delay lies on nodal points after discretization. Ravi Kanth and Murali [15] discussed an exponentially fitted spline method for singularly perturbed convection delay problems. Authors proved that the method is independent of the perturbation parameter.

In this paper, Section 2 deals with the numerical scheme for the solution of the problem on the left-end and right-end boundary layer cases. Convergence of the proposed method is discussed in Section 3. Model examples are solved using this approach and numerical results along with graphical representation are shown in Section 4. Discussions and conclusion is presented in last Section.

2. Description of the Method

Consider singularly perturbed differential-difference equation of the form:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x-\delta)+d(x)y(x)+c(x)y\left(x+\eta\right)=h(x) \tag{1}$$

subject to the interval boundary conditions

$$y(x)=\phi(x) \quad \text{on } -\delta\le x\le 0 \tag{2}$$

$$y(x)=\gamma(x), \quad \text{on } 1\le x\le 1+\eta, \tag{3}$$

where $a(x)$, $b(x)$, $c(x)$, $d(x)$, $h(x)$, $\phi(x)$ and $\gamma(x)$ are bounded and continuously differentiable functions on $(0,1)$, $0<\varepsilon\ll 1$ is the perturbation parameter; $0<\delta=o(\varepsilon)$ and $0<\eta=o(\varepsilon)$ are the delay and the advance parameters, respectively. In general, the solution of eqs. (1)–(3) exhibit the boundary layer behavior of width $O(\varepsilon)$ for small values of δ and η.

Applying Taylor series expansion in the vicinity of the point x, we have

$$y(x-\delta)\approx y(x)-\delta y'(x) \tag{4}$$

$$y\left(x+\eta\right)\approx y(x)+\eta y'(x) \tag{5}$$

Using Eq. (4) and (5) in eq. (1), we get singularly perturbed boundary value problem of the which is asymptotically equivalent to eq. (1) and is of the form:

$$\varepsilon y''(x)+p(x)y'(x)-Q(x)y(x)=h(x) \tag{6}$$

with boundary conditions

$$y(0)=\phi(0)=\phi, \tag{7}$$

$$y(1)=\gamma(1)=\gamma, \tag{8}$$

where

$$p(x)=a(x)+c(x)\eta-b(x)\delta \tag{9}$$

and

$$Q(x)=-(b(x)+c(x)+d(x)) \tag{10}$$

The conversion from eq. (1)-(3) to eq. (6)-(8) is admitted, because of the condition that $0<\delta\ll 1$ and $0<\eta\ll 1$ are sufficiently small (ref. [3]). Thus, the solution of eq. (6)-(8) will provide a good approximation to the solution of eq. (1)-(3).

Solution of eq. (6) can be described by the roots of the characteristic equation

$$\varepsilon\lambda(x)^2+p(x)\lambda(x)-Q(x)=0.$$

This equation produces two continuous functions

$$\lambda_1(x)=-\frac{p(x)}{2\varepsilon}-\sqrt{\left(\frac{p(x)}{2\varepsilon}\right)^2+\frac{Q(x)}{\varepsilon}}, \tag{11}$$

$$\lambda_2(x)=-\frac{p(x)}{2\varepsilon}+\sqrt{\left(\frac{p(x)}{2\varepsilon}\right)^2+\frac{Q(x)}{\varepsilon}}. \tag{12}$$

2.1 Left-end Boundary Layer

Discretize the domain $[0,1]$ into N equal subintervals with mesh size $h=\frac{1}{N}$, so that $x_i=x_0+ih$, $i=0,1,2,\dots,N$ are the nodes with $0=x_0$, $1=x_N$.

We consider the difference scheme for eq. (6)-(8) as:

$$\varepsilon\sigma_i(\rho)D_+D_-y_i+p(x_i)\tau_i(\rho)D_+y_i-Q(x_i)y_i=f(x_i)\quad\text{for } i=1,2,\dots,N-1 \tag{13}$$

with

$$y(0)=\phi(0)=\phi,\quad y(1)=\gamma(1)=\gamma \tag{14}$$

where $\sigma_i(\rho)$ and $\tau_i(\rho)$ are the fitting factors determined so that the solution of the corresponding homogeneous differential equation is the exact solution of the corresponding homogeneous difference eq. (13).

Here $D_+D_-y_i \approx \dfrac{y_{i-1}-2y_i+y_{i+1}}{h^2}$, $D_+y_i \approx \dfrac{y_{i+1}-y_i}{h}$ and $\rho = \dfrac{h}{\varepsilon}$.

Substituting eq. (11) and eq. (12) in the corresponding homogeneous difference equation of eq. (13), we can determine the fitting factors

$$\sigma_i(\rho) = -\frac{Q(x_i)\rho h}{4}\left(\frac{e^{-\left(\frac{p(x_i)h}{2\varepsilon}\right)}}{\sinh\left(\frac{\lambda_1(x_i)h}{2}\right)\sinh\left(\frac{\lambda_2(x_i)h}{2}\right)}\right), \quad \text{for } i = 1,2,\ldots,N-1, \tag{15}$$

and

$$\tau_i(\rho) = \frac{Q(x_i)h}{2p(x_i)}\left(\coth\left(\frac{\lambda_1(x_i)h}{2}\right)+\coth\left(\frac{\lambda_2(x_i)h}{2}\right)\right), \quad \text{for } i = 1,2,\ldots,N-1. \tag{16}$$

The tridiagonal system of the eq. (13) is

$$\left(\frac{\varepsilon\sigma_i}{h^2}\right)y_{i-1} - \left(\frac{2\varepsilon\sigma_i}{h^2}+\frac{p_i\tau_i}{h}+Q_i\right)y_i + \left(\frac{\varepsilon\sigma_i}{h^2}+\frac{p_i\tau_i}{h}\right)y_{i+1} = f_i, \quad \text{for } i = 1,2,\ldots,N-1. \tag{17}$$

The tridiagonal system eq. (17) is solved using discrete invariant imbedding algorithm using the boundary conditions eq. (14).

2.2 Right-end Boundary Layer

Discretize the domain $[0,1]$ into N equal subintervals with mesh size $h = \frac{1}{N}$, so that $x_i = x_0 + ih$, $i = 0,1,2,\ldots,N$ are the nodes with $0 = x_0$, $1 = x_N$.

We consider the difference scheme as:

$$\varepsilon\sigma_i(\rho)D_+D_-y_i + p(x_i)\tau_i(\rho)D_-y_i - Q(x_i)y_i = f(x_i), \quad \text{for } i = 1,2,\ldots,N-1 \tag{18}$$

with

$$y(0) = \phi(0) = \phi, \quad y(1) = \gamma(1) = \gamma \tag{19}$$

where $\sigma_i(\rho)$ and $\tau_i(\rho)$ are determined so that the solution of the corresponding homogeneous differential equation is the exact solution of the corresponding homogeneous difference eq. (18).

Here $D_+D_-y_i \approx \dfrac{y_{i-1}-2y_i+y_{i+1}}{h^2}$, $D_-y_i \approx \dfrac{y_i-y_{i-1}}{h}$ and $\rho = \dfrac{h}{\varepsilon}$.

Substituting eq. (11) and eq. (12) in the corresponding homogeneous difference equation of eq. (18), we can determine the fitting factors

$$\sigma_i(\rho) = -\frac{Q(x_i)\rho h}{4}\left(\frac{e^{\left(\frac{p(x_i)h}{2\varepsilon}\right)}}{\sinh\left(\frac{\lambda_1(x_i)h}{2}\right)\sinh\left(\frac{\lambda_2(x_i)h}{2}\right)}\right), \quad \text{for } i = 1,2,\ldots,N-1 \tag{20}$$

and

$$\tau_i(\rho) = \frac{Q(x_i)h}{2p(x_i)}\left(\coth\left(\frac{\lambda_1(x_i)h}{2}\right)+\coth\left(\frac{\lambda_2(x_i)h}{2}\right)\right), \quad \text{for } i = 1,2,\ldots,N-1. \tag{21}$$

The tridiagonal system of the eq. (18) is

$$\left(\frac{\varepsilon\sigma_i}{h^2}-\frac{p_i\tau_i}{h}\right)y_{i-1} - \left(\frac{2\varepsilon\sigma_i}{h^2}-\frac{p_i\tau_i}{h}+Q_i\right)y_i + \left(\frac{\varepsilon\sigma_i}{h^2}\right)y_{i+1} = f_i, \quad \text{for } i = 1,2\ldots,N-1. \tag{22}$$

The tridiagonal system eq. (22) is solved using discrete invariant imbedding algorithm using the boundary conditions eq. (19).

3. Convergence Analysis

Matrix-vector form of the tridiagonal system eq. (13) is

$$AY = C, \tag{23}$$

where $A = (m_{ij})$, $1 \le i, j \le N-1$ is a tridiagonal matrix, with

$$m_{i\,i-1} = \frac{\varepsilon\sigma}{h^2}, \quad m_{i\,i} = -\left[\frac{2\varepsilon\sigma}{h^2} + \frac{p_i\tau_i}{h} + Q_i\right], \quad m_{i\,i+1} = \frac{\varepsilon\sigma}{h^2} + \frac{p_i\tau_i}{h}$$

and $C = (d_i)$ is a column vector with $d_i = f_i$, where $i = 1,2,\dots,N-1$ with local truncation error

$$T_i(h) = h\left(\frac{\tau p_i}{2}\right)y_i'' + h^2\left(\frac{\tau p_i}{6}y_i''' + \frac{\sigma\varepsilon}{12}y_i^{iv}\right) + O(h^3) \tag{24}$$

i.e., truncation error in the difference scheme is of $O(h)$.

We also have

$$A\bar{Y} - T(h) = C, \tag{25}$$

where $\bar{Y} = (\bar{y}_0, \bar{y}_1, \dots, \bar{y}_N)^t$ denotes the actual solution and $T(h) = (T_0(h), T_1(h), \dots, T_N(h))^t$ is the local truncation error. Using eq. (23), eq. (24) and eq. (26), we get

$$A\left(\bar{Y} - Y\right) = T(h). \tag{26}$$

Thus the error equation is

$$AE = T(h) \tag{27}$$

where $E = \bar{Y} - Y = (e_0, e_1, \dots, e_N)^t$. Clearly, we have

$$S_i = \sum_{j=1}^{N-1} m_{ij} = -\left(\frac{\sigma\varepsilon}{h^2} + Q_i\right), \quad \text{for } i = 1,$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = -Q_i = B_{i_0}, \quad \text{for } i = 2,3,\dots,N-2,$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = -\left(\frac{\sigma\varepsilon}{h^2} + Q_i\right), \quad \text{for } i = N-1.$$

Since $0 < \varepsilon \ll 1$, the matrix A is irreducible and monotone. Then, it follows that A^{-1} exists and its elements are non-negative. Hence using eq. (27), we get

$$E = A^{-1}T(h) \tag{28}$$

and

$$\|E\| \le \|A^{-1}\| \cdot \|T(h)\|. \tag{29}$$

Let $\bar{m}_{k,i}$ be the (k,i)th element of A^{-1}. Since $\bar{m}_{k,i} \ge 0$, from the theory of matrices we have

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} S_i = 1, \quad k = 1,2,,\dots,N-1. \tag{30}$$

Therefore,

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} \le \frac{1}{\min\limits_{1\le i\le N-1} S_i} = \frac{1}{B_{i_o}} \le \frac{1}{|B_{i_o}|} \tag{31}$$

for some i_0 between 1 and $N-1$ and $B_{i_o} = -Q_i$.

We define

$$\|A^{-1}\| = \max_{1\le k\le N-1} \sum_{i=1}^{N-1} |\bar{m}_{ki}| \quad \text{and} \quad \|T(h)\| = \max_{1\le i\le N-1} |T_i(h)|.$$

Using eq. (23), eq. (28) and eq. (31), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{ki} T_i(h), \quad j = 1,2,\dots,N-1$$

implies

$$e_j \le \frac{kh}{|b_i|}, \quad j = 1(1)\ N-1, \tag{32}$$

where $k = \frac{\tau p_i y_i''}{4}$ is a constant.

Therefore, using eq. (32), we have

$$\|E\| = O(h)$$

i.e., our method reduces to a first order convergent on uniform mesh.

4. Numerical Experiments

Example 1. Consider problem having the boundary layer at the left-end

$$\varepsilon y'' + y' - 2y(x-\delta) - 5y + y(x+\eta) = 0$$

with boundary conditions $y(x) = 1, -\delta \le x \le 0$ and $y(x) = 1,\ 1 \le x \le 1+\eta$.

Numerical results with comparison are presented in Table 1 and Table 2. The boundary layer behaviour is shown graphically in Figure 1 and Figure 2 with different values of δ and η.

Table 1. The Maximum absolute errors in solution of Example 1

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
$\delta = 0.5\epsilon$, $\eta = 0.5\epsilon$						
Present method						
10^{-1}	4.6185e-14	1.0714e-14	2.9976e-15	3.3307e-16	1.1102e-16	1.1102e-16
10^{-2}	6.1062e-16	4.4409e-16	3.3307e-16	3.3307e-16	1.3878e-16	8.3267e-17
10^{-3}	1.6653e-15	1.3323e-15	4.4409e-16	4.4409e-16	3.8858e-16	2.2204e-16
10^{-4}	2.3037e-14	2.2427e-14	2.2315e-14	2.2038e-14	2.1982e-14	2.1538e-14
10^{-5}	3.0953e-13	3.0892e-13	3.0892e-13	3.0886e-13	3.0731e-13	2.9793e-13
10^{-6}	2.0828e-12	2.0824e-12	2.0809e-12	2.0793e-12	2.0682e-12	2.0062e-12
Results in [7]						
10^{-1}	0.0033038	0.0002201	1.29e-05	7.98e-07	4.96e-08	3.10e-09
10^{-2}	0.0235839	0.0076541	0.0031533	0.0009899	5.76e-05	3.42e-06
10^{-3}	0.0399002	0.0228969	0.0114511	0.0048135	0.0011596	0.0028588
10^{-4}	0.0418088	0.0250830	0.0137409	0.0071056	0.0035127	0.0016458
10^{-5}	0.0420041	0.0253101	0.0139842	0.0073554	0.0037635	0.0018932
10^{-6}	0.0420237	0.0253329	0.0140087	0.0073808	0.0037892	0.0019190
Results in [5]						
10^{-1}	0.1201156	0.0711396	0.0448298	0.0269461	0.0151609	0.0077503
10^{-2}	0.1872710	0.1069782	0.0590411	0.3079689	0.0156796	0.0079907
10^{-3}	0.2042972	0.1191502	0.0687923	0.0365523	0.0189384	0.0096330
10^{-4}	0.2061414	0.1204841	0.0698994	0.0372137	0.0193277	0.0098423
10^{-5}	0.2063274	0.1206188	0.0700116	0.0372808	0.0136732	0.0098636
10^{-6}	0.2063460	0.1206323	0.0700229	0.0372876	0.0193712	0.0098657

Table 2. The maximum absolute errors in solution of Example 1 with $\varepsilon = 0.1$

	$N = 8$	$N = 32$	$N = 128$	$N = 512$
Present method				
$\delta \downarrow \eta = 0.5\varepsilon$				
0.00	6.5503e-15	9.1038e-15	2.7756e-16	2.2204e-16
0.05	1.1657e-14	1.0714e-14	3.3307e-16	1.1102e-16
0.09	1.8430e-13	1.1935e-14	3.3307e-16	1.3878e-16
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	1.7747e-13	1.2546e-14	2.2204e-16	1.6653e-16
0.05	1.1657e-14	1.0714e-14	3.3307e-16	1.1102e-16
0.09	1.7952e-13	1.8874e-15	2.7756e-16	1.6653e-16
Results in [7]				
$\delta \downarrow \eta = 0.5\varepsilon$				
0.00	0.002652800	1.0220e-05	3.9258e-08	1.5344e-10
0.05	0.003303790	1.2961e-05	4.9696e-08	1.9390e-10
0.09	0.003858959	1.5323e-05	5.8654e-08	2.2897e-10
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	0.00297213	1.1559e-05	4.4365e-08	1.7319e-10
0.05	0.00330379	1.2961e-05	4.9696e-08	1.9390e-10
0.09	0.00357767	1.4124e-05	5.4113e-08	2.1112e-10
Results in [5]				
$\delta \downarrow \eta = 0.5\varepsilon$				
0.00	0.09190267	0.03453494	0.01164358	0.00300463
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.11018870	0.04110846	0.01400144	0.00362925
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	0.09720079	0.03640446	0.01229476	0.00317786
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.10632014	0.03965833	0.01348348	0.00349050

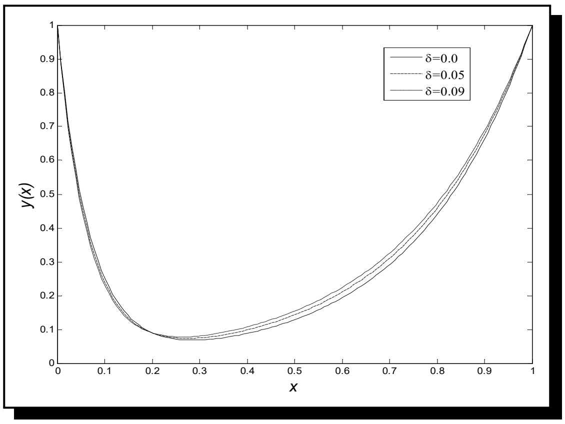

Figure 1. Numerical solution of Example 1 for different values of η with $\varepsilon = 0.1$, $\delta = 0.05$

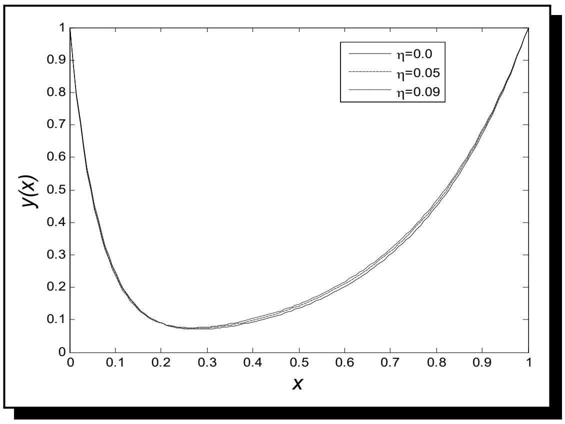

Figure 2. Numerical solution of Example 1 for different values of δ with $\varepsilon = 0.1$, $\eta = 0.05$

Example 2. Consider the boundary value problem having the boundary layer at the left end

$$\varepsilon y'' + y' - 2y(x - \delta) + y - y(x + \eta) = -1$$

with boundary conditions $y(x) = 1$, $-\delta \le x \le 0$ and $y(x) = 1$, $1 \le x \le 1 + \eta$.

Numerical solutions in comparison to the other methods are presented in Table 3. Graphical representation of the boundary layer behaviour is shown Figure 3 and Figure 4 with different values of δ and η.

Table 3. The Maximum absolute errors in solution of Example 2

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
$\delta=0.5\varepsilon$, $\eta=0.5\varepsilon$						
Present method						
10^{-1}	4.7740e-15	3.5527e-15	2.8866e-15	1.4433e-15	8.8818e-16	1.1102e-16
10^{-2}	7.6605e-15	3.2196e-15	5.5511e-16	4.4409e-16	3.3307e-16	1.1102e-16
10^{-3}	6.1062e-15	2.7756e-15	2.2204e-15	1.8874e-15	1.7764e-15	1.4433e-15
10^{-4}	2.1649e-14	2.3759e-14	2.2427e-14	2.1982e-14	2.1760e-14	2.1427e-14
10^{-5}	1.3722e-13	1.4355e-13	1.4311e-13	1.4300e-13	1.4289e-13	1.4200e-13
10^{-6}	1.9594e-12	1.9547e-12	1.9539e-12	1.9539e-12	1.9533e-12	1.9530e-12
Results in [7]						
10^{-1}	0.0004362	2.640e-05	1.58e-06	9.99e-08	6.22e-09	3.89e-10
10^{-2}	0.0059028	0.0016684	0.0030736	0.0001274	7.60e-06	4.51e-07
10^{-3}	0.0075977	0.0040202	0.0019920	0.0009022	0.00030729	0.00031863
10^{-4}	0.0007726	0.0041663	0.0021546	0.0010890	0.00054083	0.00026222
10^{-5}	0.0007738	0.0041800	0.0021689	0.0011038	0.00055620	0.00027856
10^{-6}	0.0077398	0.0041813	0.0021703	0.0011052	0.00055765	0.00028002
Results in [5]						
10^{-1}	0.0857969	0.0512956	0.0320213	0.0192472	0.0109835	0.0055359
10^{-2}	0.1337650	0.0764130	0.0421722	0.0219977	0.0111997	0.0057076
10^{-3}	0.1459266	0.0851073	0.0491373	0.0261088	0.0135274	0.0068807
10^{-4}	0.1472439	0.0860601	0.0499281	0.0265812	0.0138055	0.0070302
10^{-5}	0.1473767	0.0861563	0.0500083	0.0266292	0.0138338	0.0070454
10^{-5}	0.1473900	0.0861659	0.0500163	0.0266340	0.0138366	0.0070469

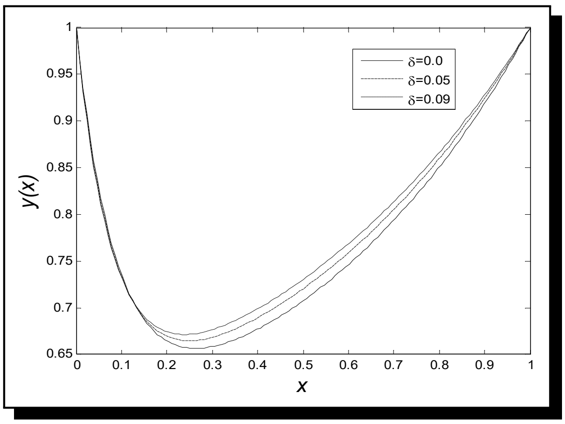

Figure 3. Numerical solution of Example 2 for different values of δ with $\varepsilon=0.1$, $\eta=0.05$

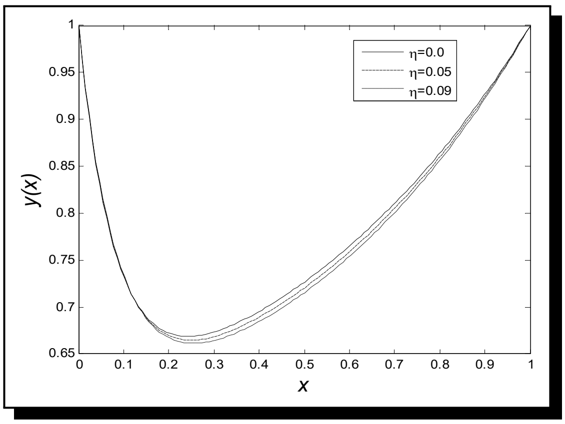

Figure 4. Numerical solution of Example 2 for different values of η with $\varepsilon=0.1$, $\delta=0.05$

Example 3. Consider the problem having the boundary layer at the right end

$$\varepsilon y''-y'-2y(x-\delta)+y-2y(x+\eta)=0$$

with boundary conditions $y(x)=1$, $-\delta\le x\le 0$ and $y(x)=-1$, $1\le x\le 1+\eta$

Computational results with comparison are presented in Table 4. The boundary layer behaviour is shown graphically in Figure 5 and Figure 6 with different values of δ and η.

Table 4. The maximum absolute errors in solution of Example 3 with $\varepsilon = 0.1$

	$N = 8$	$N = 32$	$N = 128$	$N = 512$
Present method				
$\delta \downarrow \eta = 0.5\varepsilon$				
0.00	1.4877e-14	2.7756e-15	2.1649e-15	1.1102e-16
0.05	3.6693e-13	1.8874e-15	6.1062e-16	2.2204e-16
0.09	8.3544e-15	2.1760e-14	5.5511e-16	2.7756e-16
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	1.4877e-14	2.7756e-15	2.1649e-15	1.1102e-16
0.05	3.6693e-13	1.8874e-15	6.1062e-16	2.2204e-16
0.09	3.7986e-13	2.4369e-14	5.5511e-16	1.6653e-16
Results in [7]				
$\delta \downarrow \eta = 0.5\varepsilon$				
0.00	0.002425427	8.4802e-06	3.3166e-08	1.2946e-10
0.05	0.001907515	6.7239e-06	2.6104e-08	1.0189e-10
0.09	0.001543162	5.4589e-06	2.1118e-08	8.2514e-11
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	0.001458758	5.1627e-06	1.9978e-08	7.8025e-11
0.05	0.001907515	6.7239e-06	2.6104e-08	1.0189e-10
0.09	0.002316112	8.1139e-06	3.1667e-08	1.2364e-10
Results in [5]				
$\delta \downarrow \eta = 0.5\varepsilon$				
0.00	0.09930002	0.03685072	0.01331683	0.00342882
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.10044578	0.02850398	0.01038902	0.00266379
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	0.10055269	0.02759534	0.01007834	0.00258299
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.09944067	0.03591410	0.01297367	0.00334044

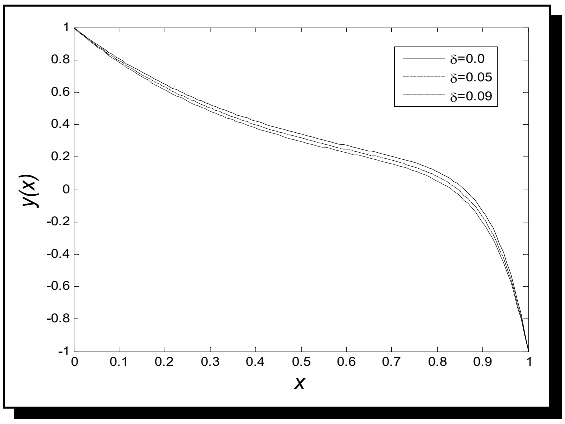

Figure 5. Numerical solution of Example 3 for different values of δ with $\varepsilon = 0.1$, $\eta = 0.05$

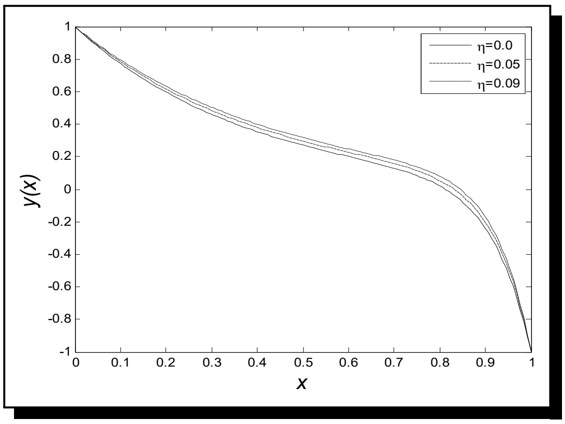

Figure 6. Numerical solution of Example 3 for different values of η with $\varepsilon = 0.1$, $\delta = 0.05$

5. Discussions and Conclusion

In this paper, a numerical scheme with two fitting factors which are inserted at convection and diffusion terms is proposed to solve singularly perturbed differential-difference equations with boundary layer behaviour. Initially, the given problem is reduced to an equivalent two point singularly perturbation problem using Taylor series on deviating terms. Then to handle the boundary layer and to get more accurate solution, two fitting factors are inserted in the scheme. Convergence of the method is analyzed. Model examples are solved using this approach and comparison with other methods in the literature is shown to justify the method. From the numerical results, we noticed that the proposed method produces very good results. Graphical representation of the boundary layer in the solution of the examples is shown in figures. Finally, we observed that the proposed is easy to implement and gives accurate results in comparison to the other methods. The proposed method is easy to implement with less computational work.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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