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Research Article

Distribution Solution of Some Nonlinear PDEs Related to the Elastic Bessel Klein-Gordon Wave Operator

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Abstract. This paper studies the existence and uniqueness of the solution to the boundary value problem for the nonlinear partial differential equation. We are particularly interested in the elastic Bessel-Helmholtz and elastic Bessel Klein-Gordon wave operators. To attain the results, the distribution theory (the generalized function theory), the iteration method and the classical Schauder estimates are applied. The solution is consequently in the form of tempered distribution (slow growth function).

Keywords. Generalized function; Tempered distribution; Elastic wave equation; Nonlinear partial differential equation

MSC. 46F10

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1. Introduction

The diamond operator, $\diamondsuit^k$, was first introduced by A. Kananthai [4]. It is defined by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial {x_i}^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial {x_j}^2}\right)^2\right]^k = \triangle^k \square^k,$$

where $\triangle^k$ and $\square^k$ are the Laplace operator iterated k times and the Ultrahyperbolic iterated k times, respectively.

Kananthai later investigated the solution of the problem [5]

$$\diamondsuit^k u(x) = f(x)$$

which is given by

$$u(x) = (-1)^k * R^e_{2k} * R^H_{2k}(x) * f(x),$$

where R^e_{2k} and $R^H_{2k}(x)$ are the fundamental solutions of $\triangle^k$ and $\square^k$, respectively.

Sritanratana and Kananthai [7] extended to the nonlinear form whose solution is related to the wave equation. While, Surikaya and Yildirim[6] explored the nonlinear structure of the Bessel diamond operator by proving the existence of the solution. They further included the constant to the operator [8].

In [2] the operators $(\triangle_B + a^2)^k$ and $(\square_B + b^2)^l$ are introduced by Bunpog and Kananthai, and named the Bessel-Helmholtz operator iterated k-times and the Bessel Klein-Gordon operator iterated k-times, respectively. They are expressed as

$$\left(\triangle_B + a^2\right)^k = \left(\sum_{i=1}^{n} B_{x_i} + a^2\right)^k$$

and

$$(\square_B + b^2)^l = \left(\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{n} B_{x_j} + b^2\right)^l,$$

where a and b are positive real numbers, n is the dimension of $\mathbb{R}^+_n$, k and l are nonnegative integers and $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$. The fundamental solutions are obtained by exploiting the properties of the Gamma function and the inverse operator.

In this paper, we study the nonlinear equation of the form

$$\left(L^{c_1}_{c_2}\right)^k_l u(x) = f\left(x, \left(L^{c_1}_{c_2}\right)^{k-1}_l u(x)\right),$$

where the continuous function f is prescribed for all $x \in \Omega \cup \partial\Omega$ in which Ω and $\partial\Omega$ denote an open subset of $\mathbb{R}^+_n$ and its boundary. The operator $\left(L^{c_1}_{c_2}\right)^k_l$ is defined by

$$\left(L^{c_1}_{c_2}\right)^k_l = \left(\triangle^{c_1}_B + a^2\right)^k \left(\square^{c_2}_B + b^2\right)^l, \quad k,l = 1,2,\dots.$$

The uniqueness of the solution is provided under the condition

$$\left| f\left(x, \left(L^{c_1}_{c_2}\right)^{k-1}_l u(x)\right)\right| \le N, \quad x \in \Omega,$$

where N is a constant and the boundary condition

$$\left(L^{c_1}_{c_2}\right)^{k-1}_l u(x) = \varphi(x), \quad x \in \partial\Omega.$$

Moreover, such a solution $u(x)$ is related to the solution of the elastic Bessel-Helmholtz and elastic Bessel Klein-Gordon wave operators.

This paper is organized into four sections. The preliminaries that include essential definitions and lemmas used for the proof of the main results are given in Section 2. Section 3 provides the main results and the conclusion is drawn in Section 4.

2. Preliminaries

Let us begin by introducing some definitions and lemmas that are occasionally referred in this paper.

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n)$, $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{R}_n^+$, c_1 be a constant. Let us define

$$Y = c_1^2(x_1^2 + x_2^2 + \dots + x_p^2) + x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2, \quad p + q = n.$$

For any complex number α, the function $S_\alpha(x)$ is represented by

$$S_\alpha(x) = \frac{2^{n+2|\nu|-2\alpha}\Gamma\left(\frac{n+2|\nu|-\alpha}{2}\right)|Y|^{\alpha-n-2|\nu|}}{\prod_{i=1}^{n} 2^{\nu_i - \frac{1}{2}}\Gamma(\nu_i + \frac{1}{2})}, \tag{2.1}$$

where Γ is the Gamma function.

Definition 2.2. Let $x = (x_1, x_2, \dots, x_n)$, $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{R}_n^+$, c_2 be a constant. The non-degenerated quadratic form is defined by

$$V = c_2^2(x_1^2 + x_2^2 + \dots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad p + q = n.$$

Furthermore, we denote the interior of the forward cone by $\Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \dots, x_n > 0, V > 0\}$ and let β be any complex number. The function $R_\beta(x)$ is expressed as

$$R_\beta(x) = \frac{V^{\frac{\beta-n-2|\nu|}{2}}}{K_n(\beta)}, \tag{2.2}$$

where

$$K_n(\beta) = \frac{\pi^{\frac{n+2|\nu|-1}{2}}\Gamma\left(\frac{2+\beta-n-2|\nu|}{2}\right)\Gamma\left(\frac{1-\beta}{2}\right)\Gamma(\beta)}{\Gamma\left(\frac{2+\beta-p-2|\nu|}{2}\right)\Gamma\left(\frac{p-\beta}{2}\right)}.$$

Definition 2.3. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_n^+$. For any complex number α, we define the function

$$T_\alpha(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{\eta}{2} + r\right)}{r!\Gamma\left(\frac{\eta}{2}\right)}(a^2)^r(-1)^{\frac{\alpha}{2}+r} S_{\alpha+2r}(x), \tag{2.3}$$

where η is a complex number and $S_{\alpha+2r}(x)$ is defined by (2.1).

Definition 2.4. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_n^+$. For any complex number β, we define the function

$$W_\beta(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{\eta}{2} + r\right)}{r!\Gamma\left(\frac{\eta}{2}\right)}(b^2)^r R_{\beta+2r}(x), \tag{2.4}$$

where β is a complex number and $R_{\beta+2r}(x)$ is given by (2.2).

Lemma 2.5. *Given the equation $(\triangle_B^{c_1} + a^2)^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $(\triangle_B^{c_1} + a^2)^k$ is the elastic Bessel-Helmholtz operator iterated k-times, defined by*

$$\left(\triangle_B^{c_1} + a^2\right)^k = \left(\frac{1}{c_1^2}\sum_{i=1}^{p} B_{x_i} + \sum_{i=p+1}^{n} B_{x_i} + a^2\right)^k, \tag{2.5}$$

where a is a positive real numbers, k is a nonnegative integers and $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}$, $x_i > 0$.

Then $u(x) = T_{2k}(x)$ is an elementary solution of such operator, where $T_{2k}(x)$ is defined by (2.3) *with $\alpha = \eta = 2k$, that is*

$$u(x) = T_{2k}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} (a^2)^r (-1)^{k+r} S_{2k+2r}(x). \tag{2.6}$$

Proof. See ([2], p.13). □

Lemma 2.6. *Given the equation $\left(\Box_B^{c2}\right)^l u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\left(\Box_B^{c2}\right)^l$ is defined by*

$$\left(\Box_B^{c2}\right)^l = \left(\frac{1}{c_2^2}\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{n} B_{x_j}\right)^l,$$

where l is a nonnegative integers and $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}, x_i > 0$. Then $u(x) = R_{2l}(x)$ where $R_{2l}(x)$ is defined by (2.2) *with $\beta = 2l$*

Proof. See [6, p. 433]. □

Lemma 2.7. *Given the equation $(\Box_B^{c2} + b^2)^l u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $(\Box_B^{c2} + b^2)^l$ is the elastic Bessel Klein-Gordon wave operator, defined by*

$$(\Box_B^{c2} + b^2)^l = \left(\frac{1}{c_2^2}\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{n} B_{x_j} + b^2\right)^l, \tag{2.7}$$

for a positive integer l. Then $u(x) = W_{2l}(x)$ is an elementary solution of such operator, where $W_{2l}(x)$ is defined by (2.4) *with $\beta = \eta = 2k$, that is*

$$u(x) = W_{2l}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(l+r)}{r!\Gamma(l)} (b^2)^r R_{2l+2r}(x).$$

Proof. First, we utilize the following formula in which its derivation is provided in ([1, p. 3]),

$$\Gamma\left(\frac{\eta}{2}+r\right) = \frac{\eta}{2}\left(\frac{\eta}{2}+1\right)\cdots\left(\frac{\eta}{2}+r-1\right)\Gamma\left(\frac{\eta}{2}\right).$$

By modifying the equation above, we obtain

$$\begin{aligned}(-1)^r \frac{1}{r!}\Gamma\left(\frac{\eta}{2}+r\right) &= \frac{(-1)^r \frac{\eta}{2}\left(\frac{\eta}{2}+1\right)\cdots\left(\frac{\eta}{2}+r-1\right)\Gamma\left(\frac{\eta}{2}\right)}{r!} \\ &= \frac{\left(-\frac{\eta}{2}\right)\left(-\frac{\eta}{2}-1\right)\cdots\left[-\left(\frac{\eta}{2}+r-1\right)\right]}{r!}\Gamma\left(\frac{\eta}{2}\right),\end{aligned}$$

which leads to

$$(-1)^r \frac{1}{r!}\Gamma\left(\frac{\eta}{2}+r\right) = \binom{-\frac{\eta}{2}}{r}\Gamma\left(\frac{\eta}{2}\right).$$

By letting $p = 2l$ in (2.4), we have

$$W_{2l}(x) = \sum_{r=0}^{\infty} \binom{-l}{r} (b^2)^r R_{2l+2r}(x).$$

Since the linear operator $\Box_B^{c2}$ is continuous and one to one mapping, then the inverse exists.

By Lemma 2.6, it gives

$$\begin{aligned} W_{2l}(x) &= \sum_{r=0}^{\infty}\binom{-l}{r}(b^2)^r(\Box_B^{c_2})^{-l-r}\delta(x) \\ &= (\Box_B^{c_2}+b^2)^{-l}\delta(x), \end{aligned} \tag{2.8}$$

where $(\Box_B^{c_2}+b^2)^{-l}$ is the inverse operator of the operator $(\Box_B^{c_2}+b^2)^{l}$. By applying the operator $(\Box_B^{c_2}+b^2)^{l}$ to both sides of (2.8), we obtain

$$(\Box_B^{c_2}+b^2)^{l}W_{2l}(x) = (\Box_B^{c_2}+b^2)^{l}.(\Box_B^{c_2}+b^2)^{-l}\delta(x).$$

Thus

$$(\Box_B^{c_2}+b^2)^{l}W_{2l}(x) = \delta(x).$$

□

Lemma 2.8. *Given the equation*

$$(\triangle_B^{c_1}+a^2)^k u(x) = f(x) \quad \text{for } \ x\in\mathbb{R}_n^+, \tag{2.9}$$

where $(\triangle_B^{c_1}+a^2)^k$ is the elastic Bessel-Helmholtz operator iterated k-times defined by (2.5) *and $f(x)$ is a given generalized function. Then $u(x)=T_{2k}(x)*f(x)$ is a solution of* (2.9)*, where $T_{2k}(x)$ is defined by* (2.6) *and k is a positive integer.*

Proof. By convolving both sides of (2.9) by $T_{2k}(x)$, we obtain

$$T_{2k}(x)*(\triangle_B^{c_1}+a^2)^k u(x) = T_{2k}(x)*f(x).$$

By the property of B-convolution and Lemma 2.5, this leads to

$$\begin{aligned} (\triangle_B^{c_1}+a^2)^k T_{2k}(x)*u(x) &= T_{2k}(x)*f(x), \\ \delta(x)*u(x) &= T_{2k}(x)*f(x), \\ u(x) &= T_{2k}(x)*f(x). \end{aligned}$$

□

Lemma 2.9. *Given the equation*

$$(\Box_B^{c_2}+b^2)^l u(x) = f(x) \quad \text{for } \ x\in\mathbb{R}_n^+, \tag{2.10}$$

where $(\Box_B^{c_2}+b^2)^l$ is the elastic Bessel Klein-Gordon wave operator iterated l-times defined by (2.7) *and $f(x)$ is a given generalized function. Then $u(x)=W_{2k}(x)*f(x)$ is a solution of* (2.10)*, where $W_{2l}(x)$ is defined by* (2.4) *and l is a positive integer.*

Proof. The proof is similar to Lemma 2.8. □

Lemma 2.10. *Given the equation*

$$(\triangle_B^{c_1}+a^2)u(x) = f(x,u(x)), \tag{2.11}$$

where $(\triangle_B^{c_1}+a^2)$ is the elastic Bessel-Helmholtz operator defined by (2.5) *with $k=1$, f is defined and has continuous first derivative for all $x\in\Omega\cup\partial\Omega$, where Ω is an open subset of $\mathbb{R}_n^+$ and $\partial\Omega$ is the boundary of Ω. Suppose that f is bounded, $\frac{\partial f}{\partial u}\geq 0$ and the boundary condition is $u(x)=\phi(x)$, then* (2.11) *has the unique solution.*

Proof. From (2.11), we have

$$\triangle_B^{c_1} u(x) = f(x,u(x)) - a^2 u(x).$$

By putting $h(x,u(x)) = f(x,u(x)) - a^2 u(x)$, we obtain

$$\triangle_B^{c_1} u(x) = h(x,u(x)).$$

By setting $u(x) = v(x) + h(x)$, where $h(x)$ satisfies $\triangle_B^{c_1} h(x) = 0$ on Ω and $h(x) = \phi(x)$ on $\partial\Omega$, the modified boundary value problem becomes

$$\triangle_B^{c_1} v(x) = h(x, v(x) + h(x)), \quad v(x) = 0 \text{ on } \partial\Omega. \tag{2.12}$$

One can prove the existence and uniqueness of the solution $v(x)$ of (2.12) by the iteration method and the Schauder estimate. The details of the proof are given by Courant and Hilbert (see [3, pp. 369 – 372]). □

Lemma 2.11. *Let $T_{2k}(x)$ and $W_{2k}(x)$ be defined by* (2.3) *and* (2.4)*, respectively. Then the convolution $T_{2k}(x) * W_{2k}(x)$ exists and it lies in $\mathcal{S}'$, where $\mathcal{S}'$ is a space of tempered distribution.*

Proof. See [2, p. 14]. □

3. Main Results

These are the main results of the paper.

Theorem 3.1. *Given the nonlinear equation*

$$\left(L_{c_2}^{c_1}\right)_l^k u(x) = f\left(x, \left(L_{c_2}^{c_1}\right)_l^{k-1} u(x)\right), \tag{3.1}$$

where $\left(L_{c_2}^{c_1}\right)_l^k$ is defined by (2.1)*. Let f have continuous first derivative for all $x \in \Omega \cup \partial\Omega$, where Ω is an open subset of $\mathbb{R}_n^+$ and $\partial\Omega$ denotes the boundary of Ω. Assume that*

$$\frac{\partial\left[\left(L_{c_2}^{c_1}\right)_l^{k-1} u\right]}{\partial u} \geq 0 \tag{3.2}$$

and f is bounded, for $x \in \Omega$

$$\left| f(x, (L_{c_2}^{c_1})_l^{k-1} u(x)) \right| \leq N, \tag{3.3}$$

where N is a constant and the boundary condition for $x \in \partial\Omega$

$$\left(L_{c_2}^{c_1}\right)_l^{k-1} u(x) = \varphi(x). \tag{3.4}$$

Then, we obtain

$$u(x) = T_{2k-2}(x) * W_{2l}(x) * F(x)$$

as a solution of (3.1)*. The refined boundary condition therefore becomes*

$$u(x) = T_{2k-2}(x) * W_{2l}(x) * \varphi(x),$$

where $F(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, $T_{2k-2}(x)$ and $W_{2l}(x)$ are defined by (2.3) *and* (2.4) *with $\alpha = 2k-2$ and $\beta = 2l$, respectively.*

Proof. From (3.1), we have

$$\left(\triangle_B^{c_1}+a^2\right)\left(L_{c_2}^{c_1}\right)_l^{k-1}u(x)=f\left(x,\left(L_{c_2}^{c_1}\right)_l^{k-1}u(x)\right). \tag{3.5}$$

Since $u(x)$ has continuous derivatives up to order $2k+2l$ for $k,\,l=1,2,\dots$, we may assume that for all $x\in\Omega$

$$\left(L_{c_2}^{c_1}\right)_l^{k-1}u(x)=F(x). \tag{3.6}$$

Thus (3.5) can be written in the form

$$\left(\triangle_B^{c_1}+a^2\right)F(x)=f(x,F(x)), \tag{3.7}$$

by (3.3) we have for $x\in\Omega$,

$$|f(x,F(x))|\le N,$$

and by (3.4) we obtain that for $x\in\partial\Omega$,

$$F(x)=\varphi(x). \tag{3.8}$$

Then, by Lemma 2.10, there exists the unique solution $F(x)$ of (3.7) which satisfies boundary condition (3.8). The equation (3.6) can be rewritten as

$$\left(\triangle_B^{c_1}+a^2\right)^{k-1}\left(\Box_B^{c_2}+b^2\right)^l u(x)=F(x). \tag{3.9}$$

By convolving both sides of (3.9) by $T_{2k-2}(x)*W_{2l}(x)$, we obtain

$$T_{2k-2}(x)*W_{2l}(x)*\left(\triangle_B^{c_1}+a^2\right)^{k-1}\left(\Box_B^{c_2}+b^2\right)^l u(x)=T_{2k-2}(x)*W_{2l}(x)*F(x). \tag{3.10}$$

By the properties of B-convolution, Lemma 2.8 and Lemma 2.9, the left hand side of (3.10) becomes

$$\delta(x)*u(x)=u(x).$$

Thus

$$u(x)=T_{2k-2}(x)*W_{2l}(x)*F(x)\quad\text{for }\ x\in\Omega.$$

Similarly, from (3.4) we obtain

$$u(x)=T_{2k-2}(x)*W_{2l}(x)*\varphi(x)\quad\text{for }\ x\in\partial\Omega,$$

which completes the proof. □

4. Conclusions

This paper focuses on finding the solution of the boundary value problem (3.1) and (3.4). We first consider the elementary solution of the elastic Bessel-Helmholtz and elastic Bessel Klein-Gordon wave operators and use them to find nonhomogeneous linear equation for such two operators. Later, we prove the existence and uniqueness of the nonlinear equation for the elastic Bessel-Helmholtz operator by utilizing the distribution theory, the iteration method and the classical Schauder estimate. Finally, we combine the previous results in order to find the solution of (3.1) and (3.4) under the conditions (3.2) and (3.3), and this leads to the solution which is in the form of the tempered distribution.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] H. Bateman, *Manuscript Project, Higher Trascendental Functions*, Vol. I, McGraw Hill, New York (1953).

[2] C. Bunpog and A. Kananthai, On the green function of the $(\diamondsuit_B + m^4)^k$ operator related to the Bessel Helmholtz operator and the Bessel Klein Gordon operator, *Journal of Applied Functional Analysis* **4**(1) (2009), 10 – 19, `http://www.eudoxuspress.com/JAFA-2009-VOL-4.pdf`.

[3] R. Courant and D. Hilbert, *Method of Mathematical Physics*, Interscience Publishers, New York (1966).

[4] A. Kananthai, On the solutions of the n-dimensional diamond operator, *Applied Mathematics and Computation* **88**(1) (1997), 27 – 37, DOI: 10.1016/S0096-3003(96)00303-7.

[5] A. Kananthai, On the diamond operator related to the wave equation, *Nonlinear Analysis: Theory, Methods & Applications* **47**(2) (2001), 1373 – 1382, DOI: 10.1016/S0362-546X(01)00272-3.

[6] M.Z. Sarikaya and H. Yildirim, On the Bessel diamond and the nonlinear Bessel diamond operator related to the Bessel wave equation, *Nonlinear Analysis: Theory, Methods & Applications* **68**(2) (2008), 430 – 442, DOI: 10.1016/j.na.2006.11.009.

[7] G. Sritanratana and A. Kananthai, On the nonlinear Diamond operator related to the wave equation, *Nonlinear Analysis: Real World Applications* **3**(4) (2002),465 – 470, DOI: 10.1016/S1468-1218(01)00027-X.

[8] H. Yildirim and M.Z. Sarikaya, On the product of the Bessel diamond and the nonlinear Bessel diamond operator related to the Bessel elastic wave equation, *Global Journal of Mathematics* **1**(1) (2015), 18 – 34, `http://gpcpublishing.org/index.php/gjm/article/view/18/0`.