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Research Article

A New Generalization of Pell-Lucas Numbers (Bi-Periodic Pell-Lucas Sequence)

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Abstract. In this study, we bring into light, a new generalization of the Jacobsthal Lucas numbers, which shall also be called the bi-periodic Jacobsthal Lucas sequence as

$$Q_n = \begin{cases} 2bQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is even} \\ 2aQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2,$$

with initial conditions $Q_0 = 2$, $Q_1 = a$. The Binet formula as well as the generating function for this sequence are given. The convergence properties of the consecutive terms of this sequence are also examined after which the well known Cassini, Catalans and the D'Ocagne's identities as well as some related summation formulas are also given.

Keywords. Bi-periodic Pell sequence; Pell-Lucas sequence; Generating function; Binet formula

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1. Introduction

Due to the numerous applications of integer sequences such as Fibonacci, Lucas, Pell, Pell-Lucas, Pell etc. in many fields of Science and Art, there have been many generalizations on them over the last century. You can see some of these different generalizations in all our references.

However, the Fibonacci and Lucas sequences have recieved a much more attention over the years than the others. It is in the light of this that we intend to write a new generalization for the Pell-Lucas sequence.

For any natural number n and any nonzero real numbers a and b, bi-periodic Fibonacci sequence, also known as the generalized Fibonacci sequence was defined recursively by Edson and Yayenie [4], and Yayenie [12] as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2$$

with initial conditions $q_0 = 0$, $q_1 = 1$, where $\lfloor a \rfloor$ is the floor function of a and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function. The Binet formula, identities such as the Cassini, Catalan and the D'Ocagne's as well as some related summation formulas were also given.

In the same way; for any natural number n and any nonzero real numbers a and b, bi-periodic Lucas sequence was defined recursively by Bilgici [1] as

$$l_n = \begin{cases} al_{n-1} + l_{n-2}, & \text{if } n \text{ is even} \\ bl_{n-1} + l_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2$$

with initial conditions $l_0 = 2$, $l_1 = a$. He also found some interesting identities between above two sequences.

In [10], Uygun and Owusu took the above generalizations one step further by defining a new generalization for the Jacobsthal sequence $\{\hat{j}_n\}_{n=0}^{\infty}$, which they called the bi-periodic Jacobsthal sequence as

$$\hat{j}_0 = 0,\ \hat{j}_1 = 1,\ \hat{j}_n = \begin{cases} a\hat{j}_{n-1} + 2\hat{j}_{n-2}, & \text{if } n \text{ is even} \\ b\hat{j}_{n-1} + 2\hat{j}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2.$$

After examining the convergence properties of the consecutive terms of this sequence, the well-known Cassini, Catalan and the D'Ocagne's identities as well as some related summation formulas were also given.

Now in this paper, just as the generalized Jacobsthal sequence and the other two mentioned above, we define a new generalization for the Pell-Lucas sequence which we shall also call bi-periodic Pell-Lucas sequence. We will then proceed to find its generating function as well as the Binet formula. The convergence properties of the consecutive terms of this sequence will be examined after which the well-known Cassini, Catalans and the D'Ocagne's identities as well as some related formulas and properties will be given.

For any two non-zero real numbers a and b, bi-periodic Pell-Lucas sequence denoted by $\{Q_n\}_{n=0}^{\infty}$ is defined recursively by

$$Q_0 = 2,\ Q_1 = 2a,\ Q_n = \begin{cases} 2bQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is even} \\ 2aQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2.$$

When $a = b = 1$, we have the classic Pell-Lucas sequence. If we set $a = b = k$, where k can be any positive number, we get the k-Pell-Lucas sequence.

The first five elements of the bi-periodic Pell-Lucas sequence are

$$Q_0 = 2,\ Q_1 = 2a,\ Q_2 = 4ab + 2,\ Q_3 = 8a^2b + 6a.$$

From the above definition we obtain the nonlinear quadratic equation for bi-periodic Pell-Lucas sequence as

$$x^2 - 2abx - ab = 0$$

with roots α and β defined by

$$\alpha = ab + \sqrt{ab(ab+1)}\,, \quad \beta = ab + \sqrt{ab(ab+1)}\,. \tag{1.1}$$

2. Main Results

Lemma 1. *The bi-periodic Pell-Lucas sequence $\{Q_n\}_{n=0}^{\infty}$ satisfies the following properties:*

- $Q_{2n} = (4ab+2)Q_{2n-2} - Q_{2n-4}\,,$
- $Q_{2n+1} = (4ab+2)Q_{2n-1} - Q_{2n-3}\,.$

Proof.

$$\begin{aligned}
Q_{2n} &= 2bQ_{2n-1} + Q_{2n-2}\\
&= 2b(2aQ_{2n-2} + Q_{2n-3}) + Q_{2n-2}\\
&= (4ab+1)Q_{2n-2} + 2bQ_{2n-3}\\
&= (4ab+1)Q_{2n-2} + (Q_{2n-2} - 2Q_{2n-4})\\
&= (4ab+2)Q_{2n-2} - Q_{2n-4}\,.
\end{aligned}$$

The other proof can be done similarly. □

Lemma 2. *α and β defined by* (1.1) *satisfy the following properties:*

- $(2\alpha+1)(2\beta+1) = 1\,,$
- $\alpha + \beta = 2ab,\ \alpha\beta = -ab\,,$
- $2\alpha + 1 = \frac{\beta^2}{ab}\,,\ 2\beta + 1 = \frac{\alpha^2}{ab}\,,$
- $-(2\alpha+1)\beta = \alpha,\ -(2\beta+1)\alpha = \beta\,.$

Proof. By using definition of α and β, the identities above can easily be proved. □

Theorem 1. *The generating function for the bi-periodic Pell-Lucas sequence is given by*

$$Q(x) = \frac{2 + 2ax - (4ab+2)x^2 + 2ax^3}{1 - (4ab+2)x^2 + x^4}\,. \tag{2.1}$$

Proof. The generating function is divided into two parts as odd part and even part

$$Q(x) = \sum_{m=0}^{\infty} Q_m x^m = Q_0(x) + Q_1(x) = \sum_{m=0}^{\infty} Q_{2m}x^{2m} + \sum_{m=0}^{\infty} Q_{2m+1}x^{2m+1}\,.$$

We simplify the even part of the above series as follows:

$$Q_0(x) = 2 + (4ab+2)x^2 + \sum_{m=2}^{\infty} Q_{2m}x^{2m}\,.$$

By multiplying through by $(4ab+2)x^2$ and x^4, respectively, we have

$$(4ab+2)x^2 Q_0(x) = (4ab+2)x^2 + (4ab+2)\sum_{m=2}^{\infty} Q_{2m-2}x^{2m}$$

and

$$x^4 Q_0(x) = \sum_{m=2}^{\infty} Q_{2m-4}x^{2m}.$$

By using Lemma 1, it is obtained that

$$\left[1-(4ab+2)x^2+x^4\right]Q_0(x) = 2-(4ab+2)x^2.$$

Hence, we get

$$Q_0(x) = \frac{2-(4ab+2)x^2}{1-(4ab+2)x^2+x^4}.$$

Similarly, the odd part of the above series is simplified as follows:

$$Q_1(x) = ax + (a^2b+6a)x^3 + \sum_{m=2}^{\infty} Q_{2m+1}x^{2m+1}.$$

By multiplying through by $(4ab+2)x^2$ and x^4, respectively, we have

$$(4ab+2)x^2 Q_1(x) = a(4ab+2)x^3 + (4ab+2)\sum_{m=2}^{\infty} Q_{2m-1}x^{2m+1}$$

and

$$x^4 Q_1(x) = \sum_{m=2}^{\infty} Q_{2m-3}x^{2m+1}.$$

By using Lemma 1, it is obtained that

$$\left[1-(4ab+2)x^2+x^4\right]Q_1(x) = ax + (a^2b+6a)x^3 - a(4ab+2)x^3 + 0.$$

Therefore,

$$Q_1(x) = \frac{ax+2ax^3}{1-(4ab+2)x^2+x^4}.$$

By adding the two results above, we obtain $Q(x)$ as

$$Q(x) = \frac{2+2ax-(4ab+2)x^2+2ax^3}{1-(4ab+2)x^2+x^4}$$

which completes the proof. □

Theorem 2. *For every n belonging to the set of natural numbers, the Binet formula for the bi-periodic Pell-Lucas sequence is given by the following:*

$$Q_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\left(\alpha^n + \beta^n\right),$$

where the parity function

$$\xi(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$.

Proof. It must be noted that the generating function for the bi-periodic Pell-Lucas sequence can also be expressed using partial fraction decomposition as:

$$Q(x)=\frac{\frac{-a}{\beta}x-\frac{1}{2\beta+1}}{x^2-(2\alpha+1)}+\frac{\frac{-a}{\alpha}x-\frac{1}{2\alpha+1}}{x^2-(2\beta+1)}.$$

The Maclaurin series expansion of the function $\frac{Ax+B}{x^2-C}$ is expressed in the form

$$\frac{Ax+B}{x^2-C}=\sum_{n=0}^{\infty}AC^{-n-1}x^{2n+1}-\sum_{n=0}^{\infty}BC^{-n-1}x^{2n}$$

and hence following the same order, the generating function $Q(x)$ can be expanded as:

$$Q(x)=\left[\sum_{n=0}^{\infty}\left\{\frac{-a}{\beta}(2\alpha+1)^{-n-1}\right\}x^{2n+1}-\sum_{n=0}^{\infty}\left\{-\left(\frac{1}{2\beta+1}\right)(2\alpha+1)^{-n-1}\right\}x^{2n}\right]$$
$$+\left[\sum_{n=0}^{\infty}\left\{\frac{-a}{\alpha}(2\beta+1)^{-n-1}\right\}x^{2n+1}-\sum_{n=0}^{\infty}\left\{-\left(\frac{1}{2\alpha+1}\right)(2\beta+1)^{-n-1}\right\}x^{2n}\right],$$

which can be expressed as

$$Q(x)=\left[-a\sum_{n=0}^{\infty}\left\{\frac{1}{\alpha}\left(\frac{1}{2\alpha+1}\right)^{n}+\frac{1}{\beta}\left(\frac{1}{2\beta+1}\right)^{n}\right\}x^{2n+1}+\sum_{n=0}^{\infty}\left\{\left(\frac{1}{2\alpha+1}\right)^{n}+\left(\frac{1}{2\beta+1}\right)^{n}\right\}x^{2n}\right]$$
$$=\left[\frac{-a}{\alpha\beta}\sum_{n=0}^{\infty}[\beta(2\beta+1)^n+\alpha(2\alpha+1)^n]x^{2n+1}+\sum_{n=0}^{\infty}[(2\beta+1)^n+(2\alpha+1)^n]x^{2n}\right].$$

By Lemma 2, it is derived that

$$Q(x)=\frac{a}{ab}\left[\sum_{n=0}^{\infty}\left\{\left[\beta\left(\frac{\beta^2}{ab}\right)^{n}+\alpha\left(\frac{\alpha^2}{ab}\right)^{n}\right]\right\}x^{2n+1}\right]+\left[\sum_{n=0}^{\infty}\{(2\beta+1)^n+(2\alpha+1)^n\}x^{2n}\right]$$
$$=\sum_{n=0}^{\infty}a\left\{\frac{\alpha^{2n+1}+\beta^{2n+1}}{(ab)^{n+1}}\right\}x^{2n+1}+\sum_{n=0}^{\infty}\left\{\left(\frac{\alpha^2}{ab}\right)^{n}+\left(\frac{\beta^2}{ab}\right)^{n}\right\}x^{2n}$$
$$=\sum_{n=0}^{\infty}a\left\{\frac{1}{(ab)^n}(\alpha^{2n-1}+\beta^{2n-1})\right\}x^{2n+1}+\left[\sum_{n=0}^{\infty}\left\{\frac{1}{(ab)^n}(\alpha^{2n}+\beta^{2n})\right\}x^{2n}\right].$$

By the help of the parity function $\xi(n)$, the above expansion can be condensed into the form

$$Q(x)=\sum_{n=0}^{\infty}\frac{a^{\xi(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}(\alpha^n+\beta^n)x^n.$$

Hence by comparing the above with the generating function $Q(x)=\sum_{n=0}^{\infty}Q_nx^n$, we the desired result is obtained as

$$Q_n=\frac{a^{\xi(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}(\alpha^n+\beta^n).$$ □

Theorem 3. *The limit of every two consecutive terms of the bi-periodic Pell-Lucas sequence is generalized as*

$$\lim_{n\to\infty}\frac{Q_{2n+1}}{Q_{2n}}=\frac{\alpha}{b},\quad \lim_{n\to\infty}\frac{Q_{2n}}{Q_{2n-1}}=\frac{\alpha}{a}.$$

Proof. Taking into account that $|\beta| < \alpha$ and $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n = 0$, we have

$$\lim_{n\to\infty}\frac{Q_{2n+1}}{Q_{2n}} = \lim_{n\to\infty}\frac{a/(ab)^{\lfloor\frac{2n+2}{2}\rfloor}}{1/(ab)^{\lfloor\frac{2n+1}{2}\rfloor}}\frac{\alpha^{2n+1}+\beta^{2n+1}}{\alpha^{2n}+\beta^{2n}} = \frac{a}{ab}\lim_{n\to\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^{2n+1}}{\frac{1}{\alpha}+\left(\frac{\beta}{\alpha}\right)^{2n+1}\frac{1}{\beta}} = \frac{\alpha}{b}.$$

The other proof can be done similarly. From this theorem we can conclude that the bi-periodic Pell-Lucas sequence does not converge. □

Theorem 4. *For any given integer n, we have*

$$Q_{-n} = (-1)^n Q_n.$$

Proof. By using Binet's formula, we get

$$\begin{aligned}
Q_{-n} &= \frac{a^{\xi(-n)}}{(ab)^{\lfloor\frac{-n+1}{2}\rfloor}}\left(1/\alpha^n + 1/\beta^n\right)\\
&= \frac{a^{\xi(n)}}{(ab)^{\lfloor\frac{-n+1}{2}\rfloor}}\frac{\beta^n+\alpha^n}{(-ab)^n}\\
&= \frac{a^{\xi(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\frac{\beta^n+\alpha^n}{(-1)^n}\\
&= (-1)^n Q_n.
\end{aligned}$$

□

Theorem 5. *Let n be any nonnegative integer, then we have*

$$\sum_{k=0}^{\infty}\binom{n}{k}2^k a^{-\xi(k)}(ab)^{\lfloor\frac{k+1}{2}\rfloor}Q_k = Q_{2n}$$

and

$$\sum_{k=0}^{\infty}\binom{n}{k}2^k a^{-\xi(k)}(ab)^{\lfloor\frac{k+1}{2}\rfloor}Q_k = Q_{2n+1}.$$

Proof. By using Binet formula and Lemma 2, we get

$$\begin{aligned}
\sum_{k=0}^{\infty}\binom{n}{k}2^k a^{-\xi(k)}(ab)^{\lfloor\frac{k+1}{2}\rfloor}Q_k &= \sum_{k=0}^{\infty}\binom{n}{k}2^k a^{-\xi(k)}(ab)^{\lfloor\frac{k+1}{2}\rfloor}\frac{a^{\xi(k)}\left(\alpha^k+\beta^k\right)}{(ab)^{\lfloor\frac{k+1}{2}\rfloor}}\\
&= \sum_{k=0}^{\infty}\binom{n}{k}\left((2\alpha)^k+(2\beta)^k\right)\\
&= (2\alpha+1)^n + (2\beta+1)^n\\
&= \left(\frac{\alpha^2}{ab}\right)^n + \left(\frac{\beta^2}{ab}\right)^n = Q_{2n}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{k=0}^{\infty}\binom{n}{k}2^k a^{\xi(k)}(ab)^{\lfloor\frac{k}{2}\rfloor}Q_{k+1} &= \sum_{k=0}^{\infty}\binom{n}{k}2^k a^{\xi(k)+\xi(k+1)}(ab)^{\lfloor\frac{k}{2}\rfloor}\frac{\left(\alpha^{k+1}+\beta^{k+1}\right)}{(ab)^{\lfloor\frac{k+2}{2}\rfloor}}\\
&= \frac{a}{ab}\sum_{k=0}^{\infty}\binom{n}{k}\left[\alpha(2\alpha)^k + \beta(2\beta)^k\right]
\end{aligned}$$

$$= \frac{\alpha(2\alpha+1)^n + \beta(2\beta+1)^n}{ab}$$
$$= a\left[\frac{\alpha}{ab}\left(\frac{\alpha^2}{ab}\right)^n + \frac{\beta}{ab}\left(\frac{\beta^2}{ab}\right)^n\right]$$
$$= a\frac{\alpha^{2n+1}+\beta^{2n+1}}{(ab)^{n+1}} = Q_{2n+1}. \qquad \square$$

Theorem 6 (Catalan Identity)**.** *For all integers n and r, Catalan Identity is given by*

$$\left(\frac{b}{a}\right)^{\xi(n+r)} Q_{n-r}Q_{n+r} - \left(\frac{b}{a}\right)^{\xi(n)} Q_n^2 = (-1)^{n-r}\left(\frac{b}{a}\right)^{\xi(r)} Q_r^2.$$

Proof. By noting the identities below, the proof proceeds as follows;

$$\xi(n+r) + \left\lfloor \frac{n-r}{2} \right\rfloor + \left\lfloor \frac{n+r}{2} \right\rfloor = n,$$
$$\xi(n+r) - \left\lfloor \frac{n-r+1}{2} \right\rfloor - \left\lfloor \frac{n+r+1}{2} \right\rfloor = -n,$$

$$I_1 = \left(\frac{b}{a}\right)^{\xi(n+r)} Q_{n-r}Q_{n+r}$$
$$= \frac{b^{\xi(n+r)}}{a^{\xi(n+r)}} a^{\xi(n-r)+\xi(n+r)} \frac{\left(\alpha^{n-r}+\beta^{n-r}\right)\left(\alpha^{n+r}+\beta^{n+r}\right)}{a^{\left\lfloor \frac{n-r+1}{2}\right\rfloor} b^{\left\lfloor \frac{n+r+1}{2}\right\rfloor}}$$
$$= \frac{a^{\xi(n-r)-\left\lfloor \frac{n-r+1}{2}\right\rfloor - \left\lfloor \frac{n+r+1}{2}\right\rfloor}}{b^{-\xi(n+r)+\left\lfloor \frac{n-r+1}{2}\right\rfloor + \left\lfloor \frac{n+r+1}{2}\right\rfloor}} \left(\alpha^{n-r}+\beta^{n-r}\right)\left(\alpha^{n+r}+\beta^{n+r}\right)$$
$$= (ab)^{-n}\left(\alpha^{2n}+\beta^{2n}+\alpha^{n-r}\beta^{n+r}+\alpha^{n+r}\beta^{n-r}\right).$$

Similarly

$$\xi(n) + 2\left\lfloor \frac{n}{2} \right\rfloor = n, \quad \xi(n) + 2\left\lfloor \frac{n+1}{2} \right\rfloor = -n,$$

$$I_2 = \left(\frac{b}{a}\right)^{\xi(n)} Q_n^2$$
$$= \frac{b^{\xi(n)}}{a^{\xi(n)}} \frac{a^{2\xi(n)}}{(ab)^{2\left\lfloor \frac{n+1}{2}\right\rfloor}} \left(\alpha^n+\beta^n\right)^2$$
$$= \frac{b^{\xi(n)-2\left\lfloor \frac{n+1}{2}\right\rfloor}}{a^{-\xi(n)+2\left\lfloor \frac{n+1}{2}\right\rfloor}} \left(\alpha^{2n}+\beta^{2n}+2\left(\alpha\beta\right)^n\right)$$
$$= \frac{b^{\xi(n)+\xi(n+1)-n-1}}{a^{-\xi(n)-\xi(n+1)+n+1}} \left(\alpha^{2n}+\beta^{2n}+2\left(\alpha\beta\right)^n\right)$$
$$= (ab)^{-n}\left(\alpha^{2n}+\beta^{2n}+2\left(\alpha\beta\right)^n\right)$$

thus

$$I_1 - I_2 = (ab)^{-n}\left[\alpha^{n-r}\beta^{n+r}+\alpha^{n+r}\beta^{n-r}+2\left(\alpha\beta\right)^n\right]$$
$$= (ab)^{-n}\left(\alpha\beta\right)^n\left[\frac{\beta^r}{\alpha^r}+\frac{\alpha^r}{\beta^r}+2\right]$$

$$
\begin{aligned}
&= \frac{(-ab)^n}{(ab)^n}\frac{\beta^{2r}+\alpha^{2r}+2\alpha^r\beta^r}{\alpha^r\beta^r} \\
&= \frac{(-ab)^n}{(ab)^n}\frac{\left(\alpha^r+\beta^r\right)^2}{(-ab)^r} \\
&= \frac{(-1)^{n-r}}{(ab)^r}\left(\alpha^r+\beta^r\right)^2 \\
&= \frac{(-1)^{n-r}}{(ab)^r}\left(\alpha^r+\beta^r\right)^2 \\
&= \frac{(-1)^{n-r}}{a^{2\xi(r)}}\frac{a^{2\xi(r)}\left(\alpha^r+\beta^r\right)^2}{(ab)^{2\lfloor\frac{r+1}{2}\rfloor-\xi(r)}} \\
&= (-1)^{n-r}\left(\frac{b}{a}\right)^{\xi(r)}Q_r^2
\end{aligned}
$$

which completes the proof □

Theorem 7 (Cassini's property or Simpson property). *For any number n belonging to the set of positive integers, we have*

$$\left(\frac{b}{a}\right)^{\xi(n+1)}Q_{n-1}Q_{n+1}-\left(\frac{b}{a}\right)^{\xi(n)}Q_n^2=(-1)^{n-1}\frac{b}{a}Q_r^2.$$

Proof. Cassini's property can easily be proven by a mere substitution of $r=1$ into the above given Catalan identity. □

Theorem 8 (D'Ocagne's property). *For any numbers m and n,belonging to the set of positive integers, with $m\geq n$, we have*

$$a^{\xi(mn+m)}b^{\xi(mn+n)}Q_mQ_{n+1}-a^{\xi(mn+m)}b^{\xi(mn+n)}Q_{m+1}Q_n=4(-1)^{n+1}(ab+1)a^{\xi(m-n)}P_{m-n}.$$

Proof. By noting the following equalities must, we proceed as follows:

$$\xi(m)+\xi(n+1)-2\xi(mn+m)=\xi(m+1)+\xi(n)-2\xi(mn+n)=1-\xi(m-n), \tag{2.2}$$

$$\xi(m-n)=\xi(mn+m)+\xi(mn+n), \tag{2.3}$$

$$\frac{m+n-\xi(m-n)}{2}=\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n+1}{2}\right\rfloor-\xi(mn+n), \tag{2.4}$$

$$\frac{m+n-\xi(m-n)}{2}=\left\lfloor\frac{m+1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor-\xi(mn+m), \tag{2.5}$$

$$\left\lfloor\frac{n+1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor=n. \tag{2.6}$$

By using the extended Binet's formula, (2.2), (2.3), (2.4), (2.5), we have

$$
\begin{aligned}
\psi &= a^{\xi(mn+n)}b^{\xi(mn+n)}Q_mQ_{n+1} \\
&= \frac{a^{\xi(mn+n)+\xi(m)+\xi(n+1)}b^{\xi(mn+n)}}{(ab)^{\lfloor\frac{m+1}{2}\rfloor+\lfloor\frac{n+2}{2}\rfloor}}\left(\alpha^m+\beta^m\right)\left(\alpha^{n+1}+\beta^{n+1}\right) \\
&= \frac{a^{\xi(mn+n)-\lfloor\frac{m}{2}\rfloor-\lfloor\frac{m+1}{2}\rfloor}}{b^{\lfloor\frac{m+1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor+1-\xi(mn+n)}}(\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{n+1}\alpha^m+\alpha^{n+1}\beta^m)
\end{aligned}
$$

$$= \frac{a^{\frac{-m-n+\xi(m-n)}{2}}}{bb^{\frac{m+n-\xi(m-n)}{2}}}(\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{n+1}\alpha^{m}+\alpha^{n+1}\beta^{m})$$
$$= \frac{1}{b(ab)^{\frac{m+n-\xi(m-n)}{2}}}(\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{n+1}\alpha^{m}+\alpha^{n+1}\beta^{m})$$

and

$$\varphi = a^{\xi(mn+m)}b^{\xi(mn+n)}Q_{m+1}Q_n$$
$$= \frac{a^{\xi(mn+m)+\xi(m+1)+\xi(n)}b^{\xi(mn+n)}}{(ab)^{\lfloor\frac{m+1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor+1}}\left(\alpha^{m+1}+\beta^{m+1}\right)\left(\alpha^{n}+\beta^{n}\right)$$
$$= \frac{a^{\xi(mn+m)-\lfloor\frac{m+1}{2}\rfloor-\lfloor\frac{n}{2}\rfloor}}{b(b)^{\frac{m+n-\xi(m-n)}{2}}}\left[\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{n}\alpha^{m+1}+\alpha^{n}\beta^{m+1}\right]$$
$$= \frac{1}{b(ab)^{\frac{m+n-\xi(m-n)}{2}}}\left[\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{n}\alpha^{m+1}+\alpha^{n}\beta^{m+1}\right].$$

From the above results, we obtain

$$\psi-\varphi = \frac{1}{b(ab)^{\frac{m+n-\xi(m-n)}{2}}}\left(\beta^{n+1}\alpha^{m}+\alpha^{n+1}\beta^{m}-\beta^{n}\alpha^{m+1}-\alpha^{n}\beta^{m+1}\right)$$
$$= \frac{1}{b(ab)^{\frac{m+n-\xi(m-n)}{2}}}\frac{-(\alpha-\beta)^{2}(\alpha\beta)^{n}\left[\alpha^{m-n}-\beta^{m-n}\right]}{\alpha-\beta}$$
$$= \frac{(-1)^{n+1}(\alpha-\beta)^{2}a^{\xi(m-n)-1}}{b}\frac{a^{1-\xi(m)}\left[\alpha^{m-n}-\beta^{m-n}\right]}{(ab)^{\lfloor\frac{m-n}{2}\rfloor}(\alpha-\beta)}$$
$$= \frac{(-1)^{n+1}(\alpha-\beta)^{2}a^{\xi(m-n)-1}}{ab}Q_{m-n} = 4(-1)^{n+1}(ab+1)a^{\xi(m-n)}(ab+8)P_{m-n}$$

which completes the proof. □

Theorem 9. *$m,n\in\mathbb{Z}^{+}$ and $n\geq m$ then* $Q_mQ_n=(ab)^{-\xi(n+m)}\left(\frac{b}{a}\right)^{-\xi(nm)}(Q_{n+m}+Q_{n-m})$.

Proof. By using the extended Binet's formula, (2.2), (2.3), (2.4), (2.5), we have

$$\left\lfloor\frac{n+m}{2}\right\rfloor=\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{m}{2}\right\rfloor+\xi(nm).$$

By using the extended Binet's formula, (2.2), (2.3), (2.4), (2.5), we have

$$Q_mQ_n = \frac{a^{\xi(m)+\xi(n)}}{(ab)^{\lfloor\frac{m+1}{2}\rfloor+\lfloor\frac{n+1}{2}\rfloor}}\left(\alpha^{m}+\beta^{m}\right)\left(\alpha^{n}+\beta^{n}\right)$$
$$= \frac{a^{m-2\lfloor\frac{m}{2}\rfloor+n-2\lfloor\frac{n}{2}\rfloor}\left(\alpha^{m+n}+\beta^{m+n}\right)}{a^{m-\lfloor\frac{m}{2}\rfloor+n-\lfloor\frac{n}{2}\rfloor}b^{m+n-\lfloor\frac{m}{2}\rfloor-\lfloor\frac{n}{2}\rfloor}}+\frac{(\alpha\beta)^{m}\left(\alpha^{n-m}+\beta^{n-m}\right)}{a^{m-\lfloor\frac{m}{2}\rfloor+n-\lfloor\frac{n}{2}\rfloor}b^{-\lfloor\frac{m}{2}\rfloor-\lfloor\frac{n}{2}\rfloor+m+n}}$$
$$= \frac{b^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor-m-n}}{a^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor}}\frac{(ab)^{\lfloor\frac{m+n+1}{2}\rfloor}}{a^{\xi(n+m)}}Q_{n+m}+\frac{(-1)^{m}(ab)^{m}b^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor-m-n}}{a^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor}}\frac{(ab)^{\lfloor\frac{n-m+1}{2}\rfloor}}{a^{\xi(n-m)}}Q_{n-m}$$
$$= \frac{b^{-\xi(nm)+\lfloor\frac{m+n}{2}\rfloor-m-n+\lfloor\frac{m+n+1}{2}\rfloor}}{a^{-\xi(nm)+\lfloor\frac{m+n}{2}\rfloor+\lfloor\frac{n}{2}\rfloor+m+n-2\lfloor\frac{m+n}{2}\rfloor-\lfloor\frac{m+n+1}{2}\rfloor}}Q_{n+m}+\frac{(-1)^{m}b^{m+\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor-m-n+\lfloor\frac{n-m+1}{2}\rfloor}}{a^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n-m+1}{2}\rfloor+\xi(n-m)-m}}Q_{n-m}$$
$$= \left(\frac{a}{b}\right)^{\xi(nm)}(Q_{n+m}-Q_{n-m}).$$

□

Theorem 10. *For any positive integer n, the first n terms of the bi-periodic of Pell Lucas sequence*

$$\sum_{k=0}^{n-1} Q_k = \frac{a^{\xi(n)} b^{1-\xi(n)} Q_n + a^{1-\xi(n)} b^{\xi(n)} Q_{n-1} - (-1)^n a}{2ab}.$$

Proof. Let n be even. By using Binet formula for bi-periodic of Pell-Lucas sequence, we get

$$\begin{aligned}\sum_{k=0}^{n-1} Q_k &= \sum_{k=0}^{\frac{n-2}{2}} Q_{2k} + \sum_{k=0}^{\frac{n-2}{2}} Q_{2k+1} \\ &= \sum_{k=0}^{\frac{n-2}{2}} \left\{ \frac{\alpha^{2k} + \beta^{2k}}{(ab)^k} + a \frac{\alpha^{2k+1} + \beta^{2k+1}}{(ab)^{k+1}} \right\}.\end{aligned}$$

If we use the property of geometric series, we get

$$= \left(\frac{\alpha^n - (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\alpha^2 - ab)} + \frac{\beta^n - (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\beta^2 - ab)} \right) + \frac{a}{ab} \left(\frac{\alpha^{n+1} - \alpha (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\alpha^2 - ab)} + \frac{\beta^{n+1} - \beta (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\beta^2 - ab)} \right).$$

After some algebraic operations, we have

$$\begin{aligned}&= \frac{1}{-4(ab)^{\frac{n}{2}+2}} \left(a^2 b^2 (\alpha^{n-2} + \beta^{n-2}) - ab(\alpha^n + \beta^n) - (ab)^{\frac{n}{2}} (\alpha^2 + \beta^2) + 2(ab)^{\frac{n}{2}+1} \right) \\ &\quad + \frac{a}{-4(ab)^{\frac{n}{2}+3}} \left(a^2 b^2 (\alpha^{n-1} + \beta^{n-1}) - ab(\alpha^{n+1} + \beta^{n+1}) + 2(ab)^{\frac{n}{2}+1} (\alpha + \beta) \right) \\ &= \frac{Q_{n-2} - Q_n - Q_2 + 2}{-4ab} + \frac{Q_{n-1} - Q_{n+1} - 4a}{-4ab} \\ &= \frac{2bQ_{n-1} + Q_2 - 2 + 2aQ_n - 4a}{4ab}.\end{aligned}$$

If n is odd, the result is in the following:

$$\begin{aligned}\sum_{k=0}^{n-1} Q_k &= \sum_{k=0}^{\frac{n-1}{2}} Q_{2k} + \sum_{k=0}^{\frac{n-3}{2}} Q_{2k+1} \\ &= \frac{-Q_{n+1} + Q_{n-1} - Q_2 + 2}{-4ab} + \frac{Q_{n-2} - Q_n + 4a}{-4ab} \\ &= \frac{2bQ_n + 2aQ_{n-1} + Q_2 - 2 - 4a}{4ab}.\end{aligned}$$

If the results combine, the desired result is obtained

$$\sum_{k=0}^{n-1} Q_k = \frac{2a^{1-\xi(n)} b^{\xi(n)} Q_n + 2a^{\xi(n)} b^{1-\xi(n)} Q_{n-1} + 4ab - 4a}{4ab}.$$ □

Theorem 11. *For any positive integer n, we have*

$$\sum_{k=0}^{n-1} \frac{Q_k}{x^k} = \frac{1}{(x^4 - (4ab+2)x^2 + 1)} \left\{ \left(\frac{Q_{n-2}}{x^{n-2}} - \frac{Q_n}{x^{n-4}} \right) + \left(\frac{Q_{n-1}}{x^{n-1}} - \frac{Q_{n+1}}{x^{n-3}} \right) + 2ax - x^2 Q_2 + 2ax^3 + 2x^4 \right\}$$

and

$$\sum_{k=0}^{\infty} \frac{Q_k}{x^k} = \frac{2ax^2 - x^2 Q_2 + 2ax^3 + 2x^4}{(x^4 - (4ab+2)x^2 + 1)}.$$

Conclusion

In this paper, bi periodic Pell Lucas sequence are defined by taking care of the parity on indices. And some very important properties of sequences such as generating function, Binet formula, Catalan Identity, D'ocagne's property, sum formulas, the limit of two consecutive terms of the bi-periodic Pell Lucas sequence are investigated. Some properties between the elements of the bi-periodic Pell Lucas sequence can also be computed.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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