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Research Article

The Boundedness of Cauchy Integral Operator on a Domain Having Closed Analytic Boundary

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Abstract. In this paper, we prove that the Cauchy integral operators (or Cauchy transforms) define continuous linear operators on the Smirnov classes for some certain domain with closed analytic boundary.

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1. Introduction

As usual, we define the Hardy space $H^2 = H^2(\Delta)$ as the space of all functions $f : z \to \sum_{n=0}^{\infty} a_n z^n$ for which the norm $\left(\|f\| = \sum_{n=0}^{\infty} |a_n|^2\right)^{1/2}$ is finite. Here, Δ is the open unit disc. For a more general simply-connected domain Ω in the complex plane $\mathbb{C}$ with at least two boundary points, and a conformal mapping φ from Ω onto Δ (that is, a Riemann mapping function), a function g analytic in Ω is said to belong to the Smirnov class $E^2(\Omega)$ if and only if $g = (f \circ \varphi)\varphi'^{1/2}$ for some $f \in H^2(\Delta)$ where $\varphi'^{1/2}$ is an analytic branch of the square root of φ'. The reader is referred to [4], [5], [8], [10], and references therein for a basic account of the subject. $\partial\Delta$ and $\partial\Omega$ will be used to denote the boundary of open unit disc Δ and the boundary of Ω, respectively.

Suppose that Γ is a simple σ-rectifiable arc (not necessarily closed). The notation $L^p(\Gamma)$ will denote the L^p space of normalized arc length measure on Γ. Let Ω denote the complement of

Γ. The Cauchy Integral of a function $\widetilde{f}$ defined on Γ and integrable relative to arc length is defined as:

$$C_\Omega \widetilde{f}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\widetilde{f}(\zeta)}{\zeta - z}d\zeta \qquad (z \in \Omega). \tag{1.1}$$

$C_\Omega \widetilde{f}$ is analytic at each point of Ω.

If Γ is not closed, then (1.1) defines a single analytic function. If Γ is closed, then Ω has two components, the interior and the exterior of Γ. Then in each component of (1.1) defines an analytic function.

Recall that a closed analytic curve is a curve $\gamma = k(\partial\Delta)$ where k is analytic and conformal in a neighbourhood U of $\partial\Delta$. If γ is simple it is called an analytic Jordan curve.

In this paper, we prove:

Theorem 1. *Suppose that that D is a bounded simply connected domain and $\gamma = \partial D$ is a closed analytic curve (e.g. ellipse). Then the Cauchy Integral*

$$C_D \widetilde{f}(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{\widetilde{f}(\zeta)}{\zeta - z}d\zeta$$

defines a continuous linear operator mapping $L^2(\partial D)$ into $E^2(D)$.

Remark 1. The result of Theorem 1 is well known in the literature (see e.g., [3], [6]). However, we give a basic and direct proof.

To prove Theorem 1, we need the following lemma and remark:

Lemma 1 ([4, p. 170])**.** *Suppose that Ω is a simply connected and bounded domain and the boundary $\Gamma = \partial\Omega$ of Ω is a rectifiable Jordan curve. Then*

(i) *Each $f \in E^2(\Omega)$ has a nontangential limit function $\widetilde{f} \in L^2(\partial\Omega)$, and*

$$\|f\|^2_{E^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\partial\Omega)} = \frac{1}{2\pi}\int_{\partial\Omega} |\widetilde{f}(z)|^2 |dz|.$$

(ii) *Each $f \in E^2(\Omega))$ has a Cauchy representation*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\widetilde{f}(\zeta)}{\zeta - z}\Omega\zeta \quad (z \in \Omega). \tag{1.2}$$

In this case, for equation (1.2), we say that Cauchy Integral Formula is valid.

A special case of the above theorem is the following remark. In fact, we prove Theorem 1 using the following remark.

Remark 2 ([11, p. 423])**.** The Cauchy integral formula

$$C_\Delta \widetilde{f}(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{\widetilde{f}(\zeta)}{\zeta - z}d\zeta$$

defines a continuous linear operator $C_\Delta : L^2(\partial\Delta) \to E^2(\Delta)$ with $\|C_\Delta\| = 1$.

There is another integral operator C_Γ on the curve Γ, which is sometimes called also Cauchy integral, viewed as an operator-valued function of the curve. This operator is given by a principal value singular integral:

if $\widetilde{f}$ is a function on Γ, we define $C_\Gamma(\widetilde{f})$ on Γ by

$$C_\Gamma\widetilde{f}(z) = \frac{1}{2\pi i}\, P.V. \int_\Gamma \frac{\widetilde{f}(\zeta)}{\zeta - z}\,d\zeta \quad (z \in \Gamma).$$

The operator C_Γ is probably less familiar than C_Ω. These operators are very important in real and complex analysis, and have attracted many mathematicians to investigate them.

In fact, there are other types of Cauchy integrals and there have been extensive literature about them and theirs applications as papers and books. For the books concerning Cauchy type integrals and related subjects (see, for instance, [1], [2], [12], [14] and [15]). For the books concerning boundedness of Cauchy type integrals (see, e.g., [7], [9] and [13]).

2. Proof of Theorem 1

Proof. Suppose that φ is a conformal map of D onto Δ. Let $\psi = \varphi^{-1} : \Delta \to D$. Consider the maps $C_\Delta : L^2(\partial\Delta) \to E^2(\Delta)$ is given by $C_\Delta\widetilde{f} = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{\widetilde{f}(\zeta)}{\zeta - z}d\zeta$ and $\widetilde{U}_\psi : L^2(\partial D) \to L^2(\partial\Delta)$ is given by $\widetilde{U}_\psi f(z) = f(\psi(z))\psi'(z)^{1/2}$ and $U_\varphi : E^2(\Delta) \to E^2(D)$ is given by $U_\varphi f(z) = f(\varphi(z))\varphi'(z)^{1/2}$. U_φ and $\widetilde{U}_\psi$ are unitary operators. The situation is illustrated in the Figure 1.

$$\begin{tikzcd}
L^2(\partial\Delta) \arrow[d, "C_\Delta"'] & L^2(\partial D) \arrow[l, "\widetilde{U}_\psi = \widetilde{U}_\varphi^*"'] \arrow[d, "C_D"] \\
E^2(\Delta) \arrow[r, "U_\varphi"] & E^2(D)
\end{tikzcd}$$

Figure 1. The maps U_φ and $\widetilde{U}_\psi$

For $\widetilde{f} \in L^2(\partial\Delta)$, we have $U_\psi C_D \widetilde{U}_\varphi \widetilde{f} = U_\psi C_D(\widetilde{f}\circ\varphi)\varphi'^{1/2}$ and

$$C_D(\widetilde{f}\circ\varphi(w))\varphi'(w)^{1/2} = \frac{1}{2\pi i}\int_{\partial D}\frac{\widetilde{f}(\varphi(\zeta))\varphi'(\zeta)^{1/2}}{\zeta - w}d\zeta$$

thus

$$\begin{aligned}
U_\psi C_D \widetilde{U}_\varphi \widetilde{f}(z) &= \psi'(z)^{1/2}\frac{1}{2\pi i}\int_{\partial D}\frac{\widetilde{f}(\varphi(\zeta))\varphi'(\zeta)^{1/2}}{\zeta - \psi(z)}d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial\Delta}\frac{\widetilde{f}(w)\psi'(w)^{1/2}\psi'(z)^{1/2}}{\psi(w) - \psi(z)}dw \qquad (z \in \Delta).
\end{aligned}$$

Then, we have kernel

$$K(z,w) = \frac{1}{w - z}H(z,w)$$

where

$$H(z,w)=\frac{(w-z)\psi'(w)^{1/2}\psi'(z)^{1/2}}{\psi(w)-\psi(z)}.$$

For any $1<r$, denote Δ_r and σ_r by $\Delta_r=\{z:|z|<1\}$ and $\sigma_r=\partial\Delta_r=\{z:|z|=r\}$.

Since ∂D is analytic (closed) curve, ψ is analytic and conformal in a neighbourhood of $\overline{\Delta}$. So without loss of generality, we may assume that ψ is analytic and conformal on Δ_R for some $R>1$. Then ψ is analytic and conformal on and inside $\sigma_{r'}$ where $R>r'>1$.

Choose r,s such that $1<r<s<R$. We shall show that H is analytic on $\Delta_r\times\Delta_r$. Hence, because r is arbitrary, it will follow that H is analytic on $\Delta_R\times\Delta_R$.

Fix $z\in\Delta_R$. Then $w\to F_z(w)=H(z,w)$ is analytic in Δ_R except at $w=z$. But since residue at $w=z$ is 0, the singularities of H for $w=z$ is removable. Hence $w\to H(z,w)$ is analytic on $\Delta_R\supseteq\overline{\Delta_s}\supseteq\overline{\Delta_r}$. We can thus apply Cauchy's integral formula to it, giving

$$H(z,w)=\frac{1}{2\pi i}\int_{\sigma_r}H(z,v)\frac{1}{v-w}dv\ \ (\text{for } w\in\Delta_r\ \text{ and fixed }\ z\in\Delta_s,\ v\in\sigma_r). \tag{2.1}$$

Hence since $z\in\Delta_s$ is arbitrary, for all $z\in\Delta_s$ and $w\in\Delta_r$, equation (2.1) is valid.

By symmetry, for every $v\in\Delta_R$ the function $z\to H(z,v)$ is analytic on Δ_R. Hence

$$H(z,v)=\frac{1}{2\pi i}\int_{\sigma_s}H(u,v)\frac{1}{u-z}du\qquad(z\in\Delta_s,v\in\Delta_R,u\in\sigma_s). \tag{2.2}$$

H is separately continuous on $\Delta_R\times\Delta_R$, so it is (jointly) continuous on $\Delta_R\times\Delta_R$. Substitute the value of $H(z,v)$ from (2.2) in the integrand of (2.1). Since the function $H(u,v)$ is continuous, we obtain

$$H(z,w)=\left(\frac{1}{2\pi i}\right)^2\int_{\sigma_r}\int_{\sigma_s}\frac{H(u,v)dudv}{(u-z)(v-w)}\qquad(z\in\Delta_s,w\in\Delta_r,v\in\sigma_r,u\in\sigma_s).$$

Since r is arbitrary, we will show that $H(z,w)$ is analytic on $\Delta_R\times\Delta_R$. Now,

$$\frac{1}{(u-z)(v-w)}=\sum_{m,n=0}^{\infty}\frac{z^m}{u^{m+1}}\frac{w^n}{v^{n+1}}\qquad(v\in\sigma_r,u\in\sigma_s)$$

and this series is uniformly convergent for $z\in\Delta_s,w\in\Delta_r,\ v\in\sigma_r,u\in\sigma_s$. Hence

$$H(z,w)=\left(\frac{1}{2\pi i}\right)^2\int_{\sigma_r}\int_{\sigma_s}\sum_{m,n=0}^{\infty}\frac{z^m}{u^{m+1}}\frac{w^n}{v^{n+1}}H(u,v)dudv\quad(z\in\Delta_s,w\in\Delta_r,v\in\sigma_r,u\in\sigma_s). \tag{2.3}$$

Since H is bounded, the series $\sum_{m,n=0}^{\infty}\frac{z^m}{u^{m+1}}\frac{w^n}{v^{n+1}}H(u,v)$ is uniformly convergent for $z\in\Delta_s,w\in\Delta_r$, $v\in\sigma_r,u\in\sigma_s$.

Because of uniformly convergence, we can integrate the series (2.3) term-by-term and we obtain

$$\begin{aligned}H(z,w)&=\sum_{m,n=0}^{\infty}z^mw^n\left(\frac{1}{2\pi i}\right)^2\int_{\sigma_r}\int_{\sigma_s}\frac{H(u,v)}{u^{m+1}v^{n+1}}dudv\quad(z\in\Delta_s,w\in\Delta_r,v\in\sigma_r,u\in\sigma_s)\\&=\sum_{m,n=0}^{\infty}a_{mn}z^mw^n\end{aligned} \tag{2.4}$$

where the coefficients a_{mn} are given by the integral formula

$$a_{mn} = \left(\frac{1}{2\pi i}\right)^2 \int_{\sigma_r}\int_{\sigma_s} \frac{H(u,v)}{u^{m+1}v^{n+1}}dudv.$$

Since H is bounded, we obtain

$$\begin{aligned}|a_{mn}| &= \left|\left(\frac{1}{2\pi i}\right)^2 \int_{\sigma_r}\int_{\sigma_s} \frac{H(u,v)}{u^{n+1}v^{m+1}}dudv\right| \\ &\le \frac{1}{4\pi^2}(2\pi s)(2\pi r)\,\|H\|_\infty \frac{1}{s^{m+1}r^{n+1}} \\ &\le s^2\,\|H\|_\infty \frac{1}{r^{m+1}r^{n+1}}\end{aligned}$$

where $\|H\|_\infty = \sup\limits_{u,v\in\sigma'} |H(u,v)| < \infty$ and so

$$\sum_{m,n=0}^{\infty} |a_{mn}| < \infty$$

(i.e. the series $\sum_{m,n=0}^{\infty} a_{mn}$ is absolutely convergent). Hence $\sum_{m,n=0}^{\infty} a_{mn}z^m w^n$ is absolutely convergent on $\Delta_r \times \Delta_r$. Thus $H(z,w)$ is analytic on $\Delta_r(\subseteq \Delta_s)\times\Delta_r$ and so since r is arbitrary it is analytic on $\Delta_R \times \Delta_R$.

Now, the series $H(z,w) = \sum_{m,n=0}^{\infty} a_{mn}z^m w^n$ is uniformly convergent for $z \in \Delta_s, w \in \Delta_r$. If we set $A = U_\psi C_D \widetilde{U}_\varphi$, then, we have

$$\begin{aligned}Af(z) &= U_\psi C_D \widetilde{U}_\varphi f(z) \qquad (f \in L^2(\partial\Delta), z \in \Delta) \\ &= \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(w)\psi'(w)^{1/2}\psi'(z)^{1/2}}{\psi(w)-\psi(z)}dw \qquad (w \in \partial\Delta) \\ &= \frac{1}{2\pi i}\int_{\partial\Delta} \frac{1}{w-z}H(z,w)f(w)dw \\ &= \frac{1}{2\pi i}\int_{\partial\Delta} \frac{1}{w-z}\sum_{m,n=0}^{\infty} a_{mn}z^m w^n f(w)dw.\end{aligned} \tag{2.5}$$

In fact, we will show that the sum and the integral in the equation (2.5) can be permutable. For $f \in L^2(\partial\Delta)$, $z \in \Delta$, we have

$$\begin{aligned}\sum_{m,n=0}^{\infty}\int_{\partial\Delta}\left|\frac{1}{w-z}a_{mn}z^m w^n f(w)\frac{dw}{2\pi i}\right| &\le \sum_{m,n=0}^{\infty}\int_{\partial\Delta}\frac{1}{|w-z|}|a_{mn}||z|^m|w|^n|f(w)|\frac{|dw|}{2\pi} \\ &\le \sum_{m,n=0}^{\infty}|a_{mn}||z|^m\left(\int_{\partial\Delta}|f(w)|^2\frac{|dw|}{2\pi}\right)^{1/2}\left(\int_{\partial\Delta}\frac{1}{|w-z|^2}\frac{|dw|}{2\pi}\right)^{1/2} \\ &\le \sum_{m,n=0}^{\infty}|a_{mn}||z|^m\,\|f\|\left(\frac{1}{(1-|z|)^2}\right)^{1/2} < \infty\end{aligned}$$

and by Tonelli Theorem

$$\frac{1}{2\pi i}\int_{\partial\Delta}\frac{1}{w-z}\sum_{m,n=0}^{\infty}a_{mn}z^m w^n f(w)dw = \sum_{m,n=0}^{\infty}a_{mn}z^m\frac{1}{2\pi i}\int_{\partial\Delta}\frac{1}{w-z}w^n f(w)dw \tag{2.6}$$

and then

$$Af(z) = \sum_{m,n=0}^{\infty}a_{mn}z^m\frac{1}{2\pi i}\int_{\partial\Delta}\frac{1}{w-z}w^n f(w)dw \qquad (f \in L^2(\partial\Delta), z \in \Delta).$$

Now consider the following series of operators

$$\sum_{m,n=0}^{\infty} a_{mn} M_m C_\Delta N_n ,$$

where $M_m : H^2(\Delta) \to H^2(\Delta)$ is defined by $M_m f(z) = z^m f(z)$ so that $\|M_m\| = \|z^m\|_\infty = 1$, and $N_n : L^2(\partial\Delta) \to L^2(\partial\Delta)$ is defined by $N_n f(z) = z^n f(z)$, so that $\|N_n\| = \|z^n\|_\infty = 1$, then, we have

$$Af(z) = \sum_{m,n=0}^{\infty} a_{mn} M_m C_\Delta N_n f(z) \qquad (f \in L^2(\partial\Delta), z \in \Delta).$$

Then the series $A = \sum_{m,n=0}^{\infty} a_{mn} M_m C_\Delta N_n$ is absolutely convergent in operator norm in the space $B(L^2(\partial\Delta), H^2(\Delta))$, in fact,

$$\begin{aligned}\|M_m C_\Delta N_n f\|_{H^2(\Delta)} &\le \|M_m\|\,\|C_\Delta\|\,\|N_n\|\,\|f\| \qquad (f \in L^2(\partial\Delta)) \\ &= \|f\|_{L^2(\partial\Delta)}\end{aligned}$$

and

$$\|M_m C_\Delta N_n\| \le 1$$

implies

$$\sum_{m,n=0}^{\infty} \|a_{mn} M_m C_\Delta N_n\| \le \sum_{m,n=0}^{\infty} |a_{mn}| < \infty$$

(i.e. $\sum_{m,n=0}^{\infty} a_{mn} M_m C_\Delta N_n$ converges absolutely) and

$$\begin{aligned}\|Af\|_{E^2(\Delta)} &= \left\| \sum_{m,n=0}^{\infty} a_{mn} M_m C_\Delta N_n f \right\|_{E^2(\Delta)} \\ &\le \sum_{m,n=0}^{\infty} \|a_{mn} M_m C_\Delta N_n f\| \qquad \left(\text{since } \sum_{m,n=0}^{\infty} a_{mn} M_m C_\Delta N_n \text{ converges absolutely}\right) \\ &= \sum_{m,n=0}^{\infty} |a_{mn}|\,\|M_m\|\,\|C_\Delta\|\,\|N_n\|\,\|f\| \\ &\le \sum_{m,n=0}^{\infty} |a_{mn}|.1.1.1\,\|f\| \\ &\le \|f\| \sum_{m,n=0}^{\infty} |a_{mn}| < \infty\end{aligned}$$

and so

$$\|A\| = \left\| \sum_{m,n=0}^{\infty} a_{mn} M_m C_\Delta N_n \right\| \le \sum_{m,n=0}^{\infty} |a_{mn}| .$$

This shows that $A = U_\psi C_D \widetilde{U}_\varphi$ is a continuous operator. It follows that C_D is a continuous operator.

Second Proof of of the Continuity of A

Since in a Banach space X (here $X = B(L^2(\partial\Delta), H^2(\Delta))$), every absolutely convergent series is convergent, in the norm of X, to an element of X, $\sum_{m,n=0}^{\infty} a_{mn} M_m C_\Delta N_n$ converges to an element

$B \in B(L^2(\partial\Delta), H^2(\Delta))$, in the sense that

$$\lim_{m,n\to\infty} \sum_{k=0}^{m}\sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l = B$$

i.e.

$$\left\| B - \sum_{k=0}^{m}\sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l \right\| \to 0 \quad \text{as } m,n\to\infty.$$

Our aim is to show that $B = A$, (i.e. $Bf(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{1}{w-z} H(z,w) f(w)dw$). Fix $z \in \Delta$ and $f \in L^2(\partial\Delta)$. Then

$$\left\| Bf - \sum_{k=0}^{m}\sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l f \right\| \to 0$$

i.e.

$$Bf = \lim_{m,n\to\infty} \sum_{k=0}^{m}\sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l f.$$

Hence

$$\sum_{k=0}^{m}\sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l f(z) \to Bf(z)$$

Now

$$\begin{aligned}
Bf(z) &= \lim_{m,n\to\infty} \sum_{k=0}^{m}\sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l f(z) \\
&= \lim_{m,n\to\infty} \frac{1}{2\pi i} \sum_{k=0}^{m}\sum_{l=0}^{n} \int_{\partial\Delta} \frac{a_{kl} z^k w^l f(w)}{w-z} dw \\
&= \lim_{m,n\to\infty} \frac{1}{2\pi i} \int_{\partial\Delta} \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{a_{kl} z^k w^l f(w)}{w-z} dw \\
&= \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(w)}{w-z} \sum_{0}^{\infty}\sum_{0}^{\infty} a_{mn} z^m w^n dw \qquad \text{(from equation (2.6))} \\
&= \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(w)}{w-z} H(z,w) dw \\
&= Af(z).
\end{aligned}$$

So for $z \in \Delta$ and $f \in L^2(\partial\Delta)$, $Bf(z) = Af(z)$. Hence $B = A$. Therefore, since B is a continuous operator it follows that $A = U_\psi C_D \widetilde{U}_\varphi$ is a continuous operator. Therefore, C_D is a continuous operator.

3. Conclusion and Further Work

In this paper we proved Theorem 1 in detail by showing the continuity of the Cauchy integral operator C_D. The result of Theorem 1 is well known in the literature, however, we give a basic and direct proof.

By using the methods of the proof of Theorem 1, further work includes investigating more complex scenarios such as choosing the domain D as (bounded or not) simply connected and ∂D, the boundary of D, as closed analytic σ-rectifiable Jordan curve. Note that an arc or closed curve

γ is called *σ-rectifiable* if and only if it is a countable union of *rectifiable* arcs in $\mathbb{C}$, together with (∞) in the case when $\infty \in \gamma$. For instance, a parabola without ∞ is *σ-rectifiable* arc, and a parabola with ∞ is *σ-rectifiable* Jordan curve.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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