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Research Article

Refinements and Reverses of Operator Callebaut Inequality Involving Tracy-Singh Products and Khatri-Rao Products

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Abstract. In this paper, we establish certain refinements and reverses of Callebaut-type inequality for bounded continuous fields of Hilbert space operators, parametrized by a locally compact Hausdorff space equipped with a finite Radon measure. These inequalities involve Tracy-Singh products, Khatri-Rao products and weighted geometric means. In addition, we obtain integral Callebaut-type inequalities for tensor products and Hadamard products. Our results extend Callebaut-type inequalities for real numbers, matrices and operators.

Keywords. Callebaut inequality; Tracy-Singh product; Khatri-Rao product; Weighted geometric mean; Continuous field of operators

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1. Introduction

In mathematics, the Cauchy-Schwarz inequality is an important inequality which can be applied in many fields, e.g. operator theory, linear algebra, analysis, probability and statistics. This inequality states that for vectors $(a_1,\dots,a_k)$ and $(b_1,\dots,b_k)$ of real numbers, we have

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right). \tag{1}$$

In 1965, Callebaut [4] published a refinement of the Cauchy-Schwarz inequality (1). For each $\alpha \in [0,1]$ and for any tuples $x = (x_1,\dots,x_k)$ and $y = (y_1,\dots,y_k)$ of positive real numbers, let us denote $\mathcal{J}_\alpha^k(x,y) = \sum_{i=1}^k x_i \sharp_\alpha y_i$, where $\sharp_\alpha$ is the α-weighted geometric mean. For either $0 \leqslant \beta \leqslant \alpha \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$, the classical Callebaut inequality [4] can be stated as

$$\left(\mathcal{J}_{1/2}^k(x,y)\right)^2 \leqslant \mathcal{J}_\alpha^k(x,y)\cdot \mathcal{J}_{1-\alpha}^k(x,y) \leqslant \mathcal{J}_\beta^k(x,y)\cdot \mathcal{J}_{1-\beta}^k(x,y) \leqslant \mathcal{J}_0^k(x,y)\cdot \mathcal{J}_1^k(x,y). \tag{2}$$

There have been several investigations and generalizations on the Callebaut inequality; see [1,2,6,7,13] and references therein. Hiai and Zhan [6] gave a matrix analogue of the Callebaut inequality (2) by considering the convexity of a certain norm function. The paper [7] presented a matrix version of (2) associated to the tensor product, the Hadamard product, weighted geometric means and a Kubo-Ando mean. Wada [13] provided a simple form of (2) for positive operators involving an operator mean and its dual. Some refinements and reverses of (2) for operators concerning the Hadamard product and weighted geometric means were presented in [1,2]. Recently in [12], the authors established integral versions of the Callebaut inequality and its refinements for bounded continuous fields of Hilbert space operators concerning the Tracy-Singh product, the Khatri-Rao product and weighted geometric means.

In this paper, we investigate refinements and reverses of the operator Callebaut inequalities for bounded continuous fields of positive operators parametrized by a locally compact Hausdorff space endowed with a finite Radon measure. Such integral inequalities involves Tracy-Singh products, Khatri-Rao products, tensor products, Hadamard products and weighted geometric means. In particular, our results are refinements and reverses of Callebaut-type inequalities obtained in the previous works [4,7,12].

This paper is organized as follows. In Section 2, we give preliminaries on operator products and Bochner integration of continuous fields of operators on a locally compact Hausdorff space. In Section 3, we provide certain refinements of integral Callebaut inequalities for bounded continuous fields of operators involving some kind of operator products and weighted geometric means. Some reversed Callebaut-type inequalities for bounded continuous fields of operators are presented in Section 4. The conclusion is given in the last section.

2. Preliminaries

Throughout this paper, let $\mathbb{H}$ be a complex Hilbert space. When $\mathbb{X}$ and $\mathbb{Y}$ are Hilbert spaces, denote by $\mathfrak{B}(\mathbb{X},\mathbb{Y})$ the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$, and abbreviate $\mathfrak{B}(\mathbb{X},\mathbb{X})$ to $\mathfrak{B}(\mathbb{X})$. For self-adjoint operators $A,B \in \mathfrak{B}(\mathbb{X})$, the notation $A \geqslant B$ means that $A-B$ is a positive operator. The set of all positive invertible operators on $\mathbb{X}$ is denoted by $\mathfrak{B}(\mathbb{X})^+$.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathbb{H} = \bigoplus_{i=1}^n \mathbb{H}_i \tag{3}$$

where all $\mathbb{H}_i$ are Hilbert spaces. For each $i = 1,\dots,n$, let P_i be the natural projection from $\mathbb{H}$ onto $\mathbb{H}_i$ and E_i the canonical embedding from $\mathbb{H}_i$ into $\mathbb{H}$. Note that $P_i^* = E_i$. Each operator

$A \in \mathfrak{B}(\mathbb{H})$ can be uniquely determined by an operator matrix

$$A = [A_{ij}]_{i,j=1}^{n,n},$$

where $A_{ij} \in \mathfrak{B}(\mathbb{H}_j, \mathbb{H}_i)$ is defined by $A_{ij} = P_i A E_j$ for each $i, j = 1, \dots, n$.

2.1 Operator Products

Recall that the tensor product of $A, B \in \mathfrak{B}(\mathbb{H})$ is a unique bounded linear operator from $\mathbb{H} \otimes \mathbb{H}$ into itself such that for all $x, y \in \mathbb{H}$,

$$(A \otimes B)(x \otimes y) = Ax \otimes By.$$

Fix a countable orthonormal basis $\mathbb{E}$ on $\mathbb{H}$. Recall that the Hadamard product of $A, B \in \mathfrak{B}(\mathbb{H})$ is defined to be bounded linear operator $A \odot B$ from $\mathbb{H}$ into itself such that for all $e \in \mathbb{E}$,

$$\langle (A \odot B)e, e \rangle = \langle Ae, e \rangle \langle Be, e \rangle.$$

Following [5], the Hadamard product can be expressed as

$$A \odot B = U^*(A \otimes B)U, \tag{4}$$

where $U : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the isometry defined by $Ue = e \otimes e$ for all $e \in \mathbb{E}$. In the case of matrices, the Hadamard product of $A = [a_{ij}]_{i,j=1}^{n,n}$ and $B = [b_{ij}]_{i,j=1}^{n,n}$ reduces to the entrywise product $A \odot B = [a_{ij} b_{ij}]$, which is a principal submatrix of the Kronecker (tensor) product $A \otimes B = [a_{ij} B]_{ij}$.

Definition 1. Let $A = [A_{ij}]_{i,j=1}^{n,n}$ and $B = [B_{ij}]_{i,j=1}^{n,n}$ be operator matrices in $\mathfrak{B}(\mathbb{H})$. The Tracy-Singh product of A and B is defined to be the operator matrix

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}, \tag{5}$$

which is a bounded linear operator from $\bigoplus_{i,j=1}^{n,n} \mathbb{H}_i \otimes \mathbb{H}_j$ into itself. The Khatri-Rao product of A and B is defined to be the operator matrix

$$A \boxdot B = [A_{ij} \otimes B_{ij}]_{i,j} \tag{6}$$

which is a bounded linear operator from $\bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{H}_i$ into itself.

Lemma 1 ([8,9])**.** *Let* $A, B, C, D \in \mathfrak{B}(\mathbb{H})$.

(1) $\alpha(A \boxtimes B) = (\alpha A) \boxtimes B = A \boxtimes (\alpha B)$ *for any* $\alpha \in \mathbb{C}$.

(2) $(A + B) \boxtimes (C + D) = A \boxtimes C + A \boxtimes D + B \boxtimes C + B \boxtimes D$.

(3) $(A \boxtimes B)^* = A^* \boxtimes B^*$.

(4) *If* $A \geqslant C \geqslant 0$ *and* $B \geqslant D \geqslant 0$*, then* $A \boxtimes B \geqslant C \boxtimes D \geqslant 0$.

(5) $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.

(6) *If* $A, B \in \mathfrak{B}(\mathbb{H})^+$*, then* $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ *for any* $\alpha \in \mathbb{R}$.

Lemma 2 ([11])**.** *There is a unital positive linear map*

$$\Phi : \mathfrak{B}\Big(\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \mathbb{H}_i \otimes \mathbb{H}_j\Big) \to \mathfrak{B}\Big(\bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{H}_i\Big) \tag{7}$$

such that $\Phi(A \boxtimes B) = A \boxdot B$ *for any* $A, B \in \mathfrak{B}(\mathbb{H})$.

2.2 Bochner Integration

Let Ω be a locally compact Hausdorff space endowed with a finite Radon measure μ. A family $(A_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ is said to be a continuous field if the parametrization $t \mapsto A_t$ is norm-continuous on Ω. If, in addition, the function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω, then we can form the Bochner integral $\int_\Omega A_t d\mu(t)$ as a unique element in $\mathfrak{B}(\mathbb{H})$ such that

$$T\Big(\int_\Omega A_t d\mu(t)\Big) = \int_\Omega T(A_t)d\mu(t) \tag{8}$$

for every T in the dual of $\mathfrak{B}(\mathbb{H})$. A field $(A_t)_{t\in\Omega}$ is said to be bounded if there is a positive constant M such that $\|A_t\| \leqslant M$ for all $t \in \Omega$. In particular, every bounded continuous field of operators on Ω is always Bochner integrable.

Lemma 3 ([10]). *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. Then for any $X \in \mathfrak{B}(\mathbb{H})$, we have*

$$\Big(\int_\Omega A_t d\mu(t)\Big) \boxtimes X = \int_\Omega (A_t \boxtimes X)d\mu(t). \tag{9}$$

3. Refined Callebaut-type Inequalities for Operators

In this section, we establish certain refined Callebaut-type inequalities for continuous fields of Hilbert space operators defined on a locally compact Hausdorff space Ω endowed with a finite Radon measure μ.

We start with recalling some auxiliary inequalities.

Lemma 4 ([2]). *Let $x, y > 0$ and $r \in (0,1)$. Then*

$$x^r y^{1-r} + x^{1-r}y^r + 2p\left(\sqrt{x}-\sqrt{y}\right)^2 + q\left(2\sqrt{xy}+x+y-2x^{\frac14}y^{\frac34}-2x^{\frac34}y^{\frac14}\right) \leqslant x+y, \tag{10}$$

where $p = \min\{r, 1-r\}$ and $q = \min\{2p, 1-2p\}$.

Lemma 5. *Decompose $\mathbb{H}$ as in (3). Let $A, B \in \mathfrak{B}(\mathbb{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac12$ or $\frac12 < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned} A^\beta \boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^\beta &\\ \geqslant A^\alpha \boxtimes B^{1-\alpha} + A^{1-\alpha}\boxtimes B^\alpha &+ \delta\Big(A^\beta\boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^\beta - 2A^{\frac12}\boxtimes B^{\frac12}\Big)\\ &+\eta\Big(A^\beta\boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^\beta + 2A^{\frac12}\boxtimes B^{\frac12} - 2A^\gamma\boxtimes B^{1-\gamma} - 2A^{1-\gamma}\boxtimes B^\gamma\Big), \end{aligned} \tag{11}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. If we replace y by x^{-1} and r by $\frac{1-u}{2}$ in (10), then we get

$$x^u + x^{-u} + 2p\left(x+x^{-1}-2\right) + q\left(x+x^{-1}+2-2x^{\frac12}-2x^{-\frac12}\right) \leqslant x + x^{-1}, \tag{12}$$

where $p = \min\left\{\frac{1-u}{2}, \frac{1+u}{2}\right\}$ and $q = \min\{2p, 1-2p\}$. Consider $v, w \in \mathbb{R}$ such that $v \leqslant w$. Applying the functional calculus on the spectrum of $A \boxtimes B$ with $u := \frac{v}{w}$ in (12), then we get

$$\begin{aligned} A^w \boxtimes B^{-w} + A^{-w}\boxtimes B^w &\\ \geqslant A^v\boxtimes B^{-v} + A^{-v}\boxtimes B^v &+ \left(\frac{w-v}{w}\right)\left(A^w\boxtimes B^{-w} + A^{-w}\boxtimes B^w - 2I\boxtimes I\right) \end{aligned}$$

$$+\eta\left(A^{w}\boxtimes B^{-w}+A^{-w}\boxtimes B^{w}+2I\boxtimes I-2A^{\frac{w}{2}}\boxtimes B^{-\frac{w}{2}}-2A^{-\frac{w}{2}}\boxtimes B^{\frac{w}{2}}\right), \tag{13}$$

where $\eta=\min\{\frac{w-v}{w},\frac{v}{w}\}$. Multiplying both sides of (13) by $A^{\frac{1}{2}}\boxtimes B^{\frac{1}{2}}$ we reach

$$\begin{aligned}&A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}\\&\geqslant A^{1+v}\boxtimes B^{1-v}+A^{1-v}\boxtimes B^{1+v}+\left(\frac{w-v}{w-1/2}\right)\left(A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}-2A\boxtimes B\right)\\&\quad+\eta\left(A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}+2A\boxtimes B-2A^{1+\frac{w}{2}}\boxtimes B^{1-\frac{w}{2}}-2A^{1-\frac{w}{2}}\boxtimes B^{1+\frac{w}{2}}\right).\end{aligned}$$

Now, we have only to replace v,w,A,B by $2\alpha-1,2\beta-1,A^{\frac{1}{2}},B^{\frac{1}{2}}$, respectively. □

Definition 2. For any bounded continuous fields $\mathcal{X}=(X_t)_{t\in\Omega}$ and $\mathcal{W}=(W_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$, we set

$$\mathcal{F}_{\mathcal{W}}(\mathcal{X})=\int_{\Omega}W_t^*X_tW_t\,d\mu(t).$$

For any bounded continuous field $\mathcal{X}=(X_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})^+$ and $\alpha\in[0,1]$, we set $\mathcal{X}^{\alpha}=\left(X_t^{\alpha}\right)_{t\in\Omega}$.

Lemma 6. *Decompose* $\mathbb{H}$ *as in* (3). *Let* $\mathcal{X}=(X_t)_{t\in\Omega}$ *and* $\mathcal{W}=(W_t)_{t\in\Omega}$ *be bounded continuous fields of operators in* $\mathfrak{B}(\mathbb{H})^+$ *and* $\mathfrak{B}(\mathbb{H})$, *respectively. If either* $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ *or* $\frac{1}{2}<\alpha\leqslant\beta\leqslant1$, *then*

$$\begin{aligned}&\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\\&\quad\geqslant\mathcal{F}_{\mathcal{W}}(\mathcal{X})^{\alpha}\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha})\\&\qquad+\delta\left[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\right]\\&\qquad+\eta\left[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})+2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\right.\\&\qquad\qquad\left.-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma})-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\right],\end{aligned} \tag{14}$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta=\min\{\delta,1-\delta\}$.

Proof. By using Lemmas 1 and 3, and Fubini's theorem for Bochner integrals [3], we get

$$\begin{aligned}\mathcal{F}_{\mathcal{W}}(\mathcal{X})^{\alpha}\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})&=\int_{\Omega}W_t^*A_t^{\alpha}W_t d\mu(t)\boxtimes\int_{\Omega}W_s^*X_s^{1-\alpha}W_s d\mu(s)\\&=\iint_{\Omega^2}\left(W_t^*X_t^{\alpha}W_t\right)\boxtimes\left(W_s^*X_s^{1-\alpha}W_s\right)\,d\mu(t)\mu(s)\\&=\iint_{\Omega^2}(W_t\boxtimes W_s)^*\left(X_t^{\alpha}\boxtimes X_s^{1-\alpha}\right)(W_t\boxtimes W_s)\,d\mu(t)\mu(s).\end{aligned}$$

We have by applying Lemma 5 that

$$\begin{aligned}&\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\\&=\iint_{\Omega^2}(W_t\boxtimes W_s)^*\left(X_t^{\beta}\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^{\beta}\right)(W_t\boxtimes W_s)\,d\mu(t)\mu(s)\\&\geqslant\iint_{\Omega^2}(W_t\boxtimes W_s)^*\left[X_t^{\alpha}\boxtimes X_s^{1-\alpha}+X_t^{1-\alpha}\boxtimes X_s^{\alpha}+\delta\left(X_t^{\beta}\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^{\beta}-2X_t^{\frac{1}{2}}\boxtimes X_s^{\frac{1}{2}}\right)\right.\\&\quad\left.+\eta\left(X_t^{\beta}\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^{\beta}+2X_t^{\frac{1}{2}}\boxtimes X_s^{\frac{1}{2}}-2X_t^{\gamma}\boxtimes X_t^{1-\gamma}-2X_t^{1-\gamma}\boxtimes X_t^{\gamma}\right)\right](W_t\boxtimes W_s)\,d\mu(t)\mu(s)\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{F}_{\mathcal{W}}(\mathcal{X})^{\alpha} \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \\
&\quad + \delta\Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) - 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\Big] \\
&\quad + \eta\Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta}) + 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \\
&\qquad - 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) - 2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma}) \boxtimes \mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\Big]. \qquad \square
\end{aligned}
$$

Recall that, for each $\alpha \in [0,1]$, the α-weighted geometric mean of operators $X, Y \in \mathfrak{B}(\mathbb{H})^+$ is defined as

$$X \sharp_{\alpha} Y = X^{\frac{1}{2}}\big(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}}\big)^{\alpha} X^{\frac{1}{2}}.$$

Definition 3. For two continuous fields $\mathcal{A} = (A_t)_{t\in\Omega}$, $\mathcal{B} = (B_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})^+$ and any $\alpha \in [0,1]$, we set

$$\mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) = \int_{\Omega} A_t \sharp_{\alpha} B_t \, d\mu(t).$$

In particular, we have

$$\mathcal{I}_0(\mathcal{A},\mathcal{B}) = \int_{\Omega} A_t \, d\mu(t), \quad \mathcal{I}_1(\mathcal{A},\mathcal{B}) = \int_{\Omega} B_t \, d\mu(t).$$

Consider bounded continuous fields $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})^+$. An integral Callebaut inequality [12] states that for either $0 \leqslant \beta \leqslant \alpha \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$, we have

$$
\begin{aligned}
2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) &\leqslant \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \\
&\leqslant \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \\
&\leqslant \mathcal{I}_{0}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{0}(\mathcal{A},\mathcal{B}). \qquad (15)
\end{aligned}
$$

Now, we provide a refinement of the integral Callebaut inequality (15) and as a consequence give an operator Callebaut type inequality for Khatri-Rao products.

Theorem 1. *Decompose $\mathbb{H}$ as in* (3)*. Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$
\begin{aligned}
&\mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \\
&\quad \geqslant \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \\
&\qquad + \delta\Big[\mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big] \\
&\qquad + \eta\Big[\mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \\
&\qquad - 2\mathcal{I}_{\gamma}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\gamma}(\mathcal{A},\mathcal{B})\Big], \qquad (16)
\end{aligned}
$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. Setting $X_t = A_t^{-\frac{1}{2}} B_t A_t^{-\frac{1}{2}}$ and $W_t = A_t^{\frac{1}{2}}$ for all $t \in \Omega$, we have that for any $\alpha \in [0,1]$,

$$\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha}) = \int_{\Omega} A_t^{\frac{1}{2}}\Big(A_t^{-\frac{1}{2}} B_t A_t^{-\frac{1}{2}}\Big)^{\alpha} A_t^{\frac{1}{2}} \, d\mu(t) = \int_{\Omega} A_t \sharp_{\alpha} B_t \, d\mu(t) = \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}).$$

By using Lemma 6, we obtain the result. $\square$

Corollary 1. *Decompose $\mathbb{H}$ as in* (3)*. Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \\
&\quad \geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \\
&\qquad + \delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big] \\
&\qquad + \eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \\
&\qquad\qquad - 2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\Big],
\end{aligned} \tag{17}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. Let Φ be the linear map described in Lemma 2. We have that for any $\alpha \in [0,1]$,

$$\Phi(\mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})) = \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \boxdot \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}).$$

The proof is done by using Theorem 1 and the fact that the map Φ is a positive unital linear map. □

The next result is an integral inequality involving tensor products which is a special case of Theorem 1 when $n = 1$.

Corollary 2. *Let $\mathbb{H}$ be a Hilbert space (not decomposed as in* (3)*). Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \\
&\quad \geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \\
&\qquad + \delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big] \\
&\qquad + \eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\beta(\mathcal{A},\mathcal{B}) + 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \\
&\qquad\qquad - 2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B}) \otimes \mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\Big],
\end{aligned} \tag{18}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

As a consequence, we obtain the following integral inequality concerning Hadamard products.

Corollary 3. *Let $\mathbb{H}$ be a Hilbert space. Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \\
&\quad \geqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \delta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) - \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\Big] \\
&\qquad\qquad + \eta\Big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) - 2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B}) \odot \mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\Big],
\end{aligned} \tag{19}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. Using the fact that the Hadamard product is expressed as the deformation of the tensor product via the isometry U defined in (4), we get the result. $\square$

Remark 1. When we set $\Omega = \{1,\dots,k\}$ equipped with the counting measure, we have that $\mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) = \sum_{i=1}^{k} A_i \sharp_\alpha B_i$. From previous theorem and previous corollaries, we obtain discrete versions of refined Callebaut-type inequalities for Tracy-Singh products, Khatri-Rao products, tensor products and Hadamard products, respectively.

Remark 2. For a particular case of Theorem 1 when $\mathbb{H} = \mathbb{C}^n$ and $\Omega = \{1,\dots,k\}$ equipped with the counting measure, we get a matrix inequality concerning Tracy-Singh products. In the same way, we get matrix versions of (17)-(19) for Khatri-Rao products, Kronecker products and Hadamard products, respectively. The matrix versions of Kronecker products and Hadamard products are refinements of matrix Callebaut inequalities in [7, Theorem 3.4 and Corollary 3.5].

In the next corollary, we get a refined Callebaut-type inequality for real numbers.

Corollary 4. *Let $x = (x_t)_{t\in\Omega}$ and $y = (y_t)_{t\in\Omega}$ be two fields of positive real numbers. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y) \geqslant{}& \mathcal{I}_\alpha(x,y)\cdot\mathcal{I}_{1-\alpha}(x,y) + \delta\Big[\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y) - \big(\mathcal{I}_{1/2}(x,y)\big)^2\Big]\\ &+\eta\Big[\mathcal{I}_\beta(x,y)\cdot\mathcal{I}_{1-\beta}(x,y) + \big(\mathcal{I}_{1/2}(x,y)\big)^2 - 2\mathcal{I}_\gamma(x,y)\cdot\mathcal{I}_{1-\gamma}(x,y)\Big],\end{aligned}$$

where $\gamma = \frac{1+2\beta}{4}$, $\delta = \frac{\beta-\alpha}{\beta-1/2}$ and $\eta = \min\{\delta, 1-\delta\}$.

Proof. Putting $A_t = x_t I$ and $B_t = y_t I$ for all $t \in \Omega$ in Theorem 1, we obtain the result. $\square$

We mention that if Ω is the finite set $\{1,\dots,k\}$ equiped with the counting measure, we get a discrete version of (20) which is a refinement of the classical Callebaut inequality (2).

4. Reversed Callebaut-type Inequalities for Operators

In this section, we present reversed inequalities of Callebaut-type inequalities. We begin with recalling the following scalar inequality.

Lemma 7 ([14]). *Let $x, y \geqslant 0$ and $r \in (0,1)$.*

$$x + y \leqslant x^{1-r}y^r + x^r y^{1-r} + 2s(\sqrt{x}-\sqrt{y})^2 - q\left(x+y+2\sqrt{xy} - 2x^{\frac{1}{4}}y^{\frac{3}{4}} - 2x^{\frac{3}{4}}y^{\frac{1}{4}}\right),$$

where $p = \min\{r, 1-r\}$, $q = \min\{2p, 1-2p\}$ and $s = \max\{r, 1-r\}$.

This lemma is used to derive the following operator inequality.

Lemma 8. *Decompose $\mathbb{H}$ as in (3). Let $A, B \in \mathfrak{B}(\mathbb{H})^+$ and either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$. Then*

$$\begin{aligned}&A^\beta \boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^\beta\\ &\quad\leqslant A^\alpha\boxtimes B^{1-\alpha} + A^{1-\alpha}\boxtimes B^\alpha + (2-\delta)\left(A^\beta\boxtimes B^{1-\beta} + A^{1-\beta}\boxtimes B^\beta - 2A^{\frac{1}{2}}\boxtimes B^{\frac{1}{2}}\right)\end{aligned}$$

$$-\eta\left(A^{\beta}\boxtimes B^{1-\beta}+A^{1-\beta}\boxtimes B^{\beta}+2A^{\frac{1}{2}}\boxtimes B^{\frac{1}{2}}-2A^{\gamma}\boxtimes B^{1-\gamma}-2A^{1-\gamma}\boxtimes B^{\gamma}\right), \tag{20}$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta=\min\{\delta,1-\delta\}$.

Proof. In Lemma 7, we have by replacing y with x^{-1} that

$$x+x^{-1}\leqslant x^{1-2r}+x^{2r-1}+2s(x+x^{-1}-2)-q(x+x^{-1}+2-2x^{-\frac{1}{2}}-2x^{\frac{1}{2}}).$$

Let $u\in(0,1]$. Taking $r=\frac{1-u}{2}$, we obtain

$$x+x^{-1}\leqslant x^{u}+x^{-u}+(1+u)(x+x^{-1}-2)-q\left(x+x^{-1}+2-2x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\right).$$

Consider real numbers v,w such that $\frac{v}{w}\in(0,1]$. Using the functional calculus on the spectrum of $A\boxtimes B$ and Lemma 1, and putting $u=\frac{v}{w}$, we get

$$\begin{aligned}&A^{w}\boxtimes B^{-w}+A^{-w}\boxtimes B^{w}\\&\quad\leqslant A^{v}\boxtimes B^{-v}+A^{-v}\boxtimes B^{v}+\left(1+\frac{v}{w}\right)(A^{w}\boxtimes B^{-w}+A^{-w}\boxtimes B^{w}-2I\boxtimes I)\\&\qquad-\eta\left(A^{w}\boxtimes B^{-w}+A^{-w}\boxtimes B^{w}+2I\boxtimes I-2A^{\frac{w}{2}}\boxtimes B^{-\frac{w}{2}}-2A^{-\frac{w}{2}}\boxtimes B^{\frac{w}{2}}\right).\end{aligned}$$

Multiplying both sides by $A^{\frac{1}{2}}\boxtimes B^{\frac{1}{2}}$ and applying Lemma 1, we have

$$\begin{aligned}&A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}\\&\quad\leqslant A^{1+v}\boxtimes B^{1-v}+A^{1-v}\boxtimes B^{1+v}+\left(1+\frac{v}{w}\right)(A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}-2A\boxtimes B)\\&\qquad-\eta\left(A^{1+w}\boxtimes B^{1-w}+A^{1-w}\boxtimes B^{1+w}+2A\boxtimes B-2A^{1+\frac{w}{2}}\boxtimes B^{1-\frac{w}{2}}-2A^{1-\frac{w}{2}}\boxtimes B^{1+\frac{w}{2}}\right),\end{aligned}$$

where $\gamma=\min\left\{\frac{v}{w},1-\frac{v}{w}\right\}$. We reach the result by replace v,w,A,B with $2\alpha-1,2\beta-1,A^{\frac{1}{2}},B^{\frac{1}{2}}$, respectively. □

Lemma 9. *Decompose* $\mathbb{H}$ *as in* (3). *Let* $\mathcal{X}=(X_t)_{t\in\Omega}$ *and* $\mathcal{W}=(W_t)_{t\in\Omega}$ *be bounded continuous fields of operators in* $\mathfrak{B}(\mathbb{H})^{+}$ *and* $\mathfrak{B}(\mathbb{H})$, *respectively. If either* $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ *or* $\frac{1}{2}<\alpha\leqslant\beta\leqslant1$, *then*

$$\begin{aligned}&\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}_t^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}_t^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\\&\leqslant\mathcal{F}_{\mathcal{W}}(\mathcal{X}_t^{\alpha})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\alpha})\\&\quad+(2-\delta)\left[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\right]\\&\quad-\eta\Big[\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\beta})+2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\\&\quad-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma})-2\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{1-\gamma})\boxtimes\mathcal{F}_{\mathcal{W}}(\mathcal{X}^{\gamma})\Big],\end{aligned} \tag{21}$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ *and* $\eta=\min\{\delta,1-\delta\}$.

Proof. The proof is similar to that of Lemma 6. Instead of using Lemma 5, we apply Lemma 8. □

The next theorem is a reverse of the second inequality of (15) involving Tracy-Singh products. As a consequence, we get a reversed Callebaut-type inequality for Khatri-Rao products by using the unital positive linear map Φ in Lemma 2.

Theorem 2. *Decompose $\mathbb{H}$ as in* (3)*. Let $\mathcal{A}=(A_t)_{t\in\Omega}$ and $\mathcal{B}=(B_t)_{t\in\Omega}$ be two bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$. If either $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ or $\frac{1}{2}<\alpha\leqslant\beta\leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+(2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big]\\
&\qquad-\eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big],
\end{aligned}$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ and $\eta=\min\{\delta,1-\delta\}$.

Proof. The proof is similar to that of Theorem 1. Instead of using Lemma 6, we apply Lemma 9. □

Corollary 5. *Under the same hypothesis and notation as in Theorem* (2)*, we have*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+(2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big]\\
&\qquad-\eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\boxdot\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big].
\end{aligned}$$

For the case $n=1$ (i.e. $\mathbb{H}$ is not decomposed), Theorem 2 reduces to the reversed Callebaut-type inequality for tensor products and consequently applies to Hadamard products as follows.

Corollary 6. *Under the same hypothesis and notation as in Theorem* 2 *except that the Hilbert space $\mathbb{H}$ is not decomposed, we have*

$$\begin{aligned}
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\\
&\qquad+(2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big]\\
&\qquad-\eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\beta(\mathcal{A},\mathcal{B})+2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\\
&\qquad-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})-2\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\otimes\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\big],\\
&\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\\
&\quad\leqslant \mathcal{I}_\alpha(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+(2-\delta)\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})-\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big]\\
&\qquad-\eta\big[\mathcal{I}_\beta(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})-2\mathcal{I}_\gamma(\mathcal{A},\mathcal{B})\odot\mathcal{I}_{1-\gamma}(\mathcal{A},\mathcal{B})\big].
\end{aligned}$$

Remark 3. From previous results, we obtain discrete versions of reversed Callebaut-type inequalities for Tracy-Singh products, Khatri-Rao products, tensor products and Hadamard products, respectively, by setting $\Omega=\{1,\dots,k\}$ equiped with the counting measure. Matrix analogues of our results can be obtained particularly by setting $\mathbb{H}=\mathbb{C}^n$. Such matrix results of Kronecker products and Hadamard products are reverses of the matrix Callebaut inequalities in [7, Theorem 3.4 and Corollary 3.5].

The following corollary is a reverse of the integral Callebaut inequality for real numbers. In particular, when Ω is the finite set $\{1,\dots,k\}$ equiped with the counting measure, we get a reversed inequality of the second inequality of (2).

Corollary 7. *Let $x=(x_t)_{t\in\Omega}$ and $y=(y_t)_{t\in\Omega}$ be two fields of positive real numbers. If either $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ or $\frac{1}{2}<\alpha\leqslant\beta\leqslant1$, then*

$$\begin{aligned}\mathcal{I}_{\beta}(x,y)\cdot\mathcal{I}_{1-\beta}(x,y)\leqslant\mathcal{I}_{\alpha}(x,y)\cdot\mathcal{I}_{1-\alpha}(x,y)&+(2-\delta)\left[\mathcal{I}_{\beta}(x,y)\cdot\mathcal{I}_{1-\beta}(x,y)-\left(\mathcal{I}_{1/2}(x,y)\right)^{2}\right]\\&-\eta\left[\mathcal{I}_{\beta}(x,y)\cdot\mathcal{I}_{1-\beta}(x,y)+\left(\mathcal{I}_{1/2}(x,y)\right)^{2}-2\mathcal{I}_{\gamma}(x,y)\cdot\mathcal{I}_{1-\gamma}(x,y)\right],\end{aligned}$$

where $\gamma=\frac{1+2\beta}{4}$, $\delta=\frac{\beta-\alpha}{\beta-1/2}$ and $\eta=\min\{\delta,1-\delta\}$.

5. Conclusion

We establish certain refinements and reverses of Callebaut-type inequalities for bounded continuous fields of operators which are parametrized by a locally compact Hausdorff space Ω equipped with a finite Radon measure. These inequalities involve Tracy-Singh products, Khatri-Rao products, tensor products, Hadamard products and weighted geometric means. When Ω is a finite space equipped with the counting measure, such integral inequalities reduce to discrete inequalities. Our results include matrix results concerning the Tracy-Singh product, the Khatri-Rao product, the Kronecker product, and the Hadamard product. In particular, we get a refinement and a reverse of the classical Callebaut inequality for real numbers.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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