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Wavelets and Frames Based on Walsh-Dirichlet Type Kernels

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Abstract. Using the Walsh-Dirichlet kernel and some of its modifications, we construct several examples of frames and periodic wavelets on the positive half-line. The corresponding algorithms for decomposition and reconstruction are also discussed. It is noted that similar results can be obtained for wavelets and frames on the Cantor and Vilenkin groups.

1. **Introduction**

Orthogonal wavelets and refinable functions representable as lacunary Walsh series have been initiated in [13]; recent results in this direction can be found in [3]-[7] and the references therein. In the present paper, by analogy with frames and wavelets on the line $\mathbb{R}$, which are usually determined by appropriately chosen trigonometric or orthogonal polynomials (cf. [2], [8] [15]), we study frames and periodic wavelets on the positive half-line $\mathbb{R}_+$ associated with the Walsh-Dirichlet kernel and some of its modifications. Results on mappings into the Walsh polynomial spaces and algorithms for decomposition and reconstruction are also discussed. We note that similar results can be obtained in a more general setting, e.g., for wavelets and frames on the Cantor and Vilenkin groups (cf. [13], [4]).

Let us recall that the *Walsh system* $\{w_l \mid l \in \mathbb{Z}_+\}$ on $\mathbb{R}_+$ is defined as

$$w_0(x) \equiv 1, \qquad w_l(x) = \prod_{j=0}^{k} (w_1(2^j x))^{\nu_j}, \quad l \in \mathbb{N}, \quad x \in \mathbb{R}_+,$$

where k and ν_j are deduced from the dyadic expansion

$$l = \sum_{j=0}^{k} \nu_j 2^j, \quad \nu_j \in \{0,1\}, \quad \nu_k = 1, \quad k = k(l),$$

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and $w_1(x)$ is defined on $[0,1)$ by the formula

$$w_1(x) = \begin{cases} 1, & x \in [0,1/2), \\ -1, & x \in [1/2,1) \end{cases}$$

and is extended to $\mathbb{R}_+$ such that $w_1(x+1) = w_1(x)$ for all $x \in \mathbb{R}_+$. For basic properties of the Walsh series and their numerical applications see, e.g., [9], [10], [16], [17].

We shall denote the integer and the fractional parts of a number x by $[x]$ and $\{x\}$, respectively. For each $x \in \mathbb{R}_+$ it is possible to take x_j, $x_{-j} \in \{0,1\}$ such that

$$x = [x] + \{x\} = \sum_{j=1}^{\infty} x_{-j} 2^{j-1} + \sum_{j=1}^{\infty} x_j 2^{-j}$$

(for a dyadic rational x we chose an expansion with finitely many nonzero terms). It is easy to see that

$$x_{-j} = [2^{1-j}x] (\bmod 2) \quad \text{and} \quad x_j = [2^j x] (\bmod 2) \quad \text{for all } j \in \mathbb{N}.$$

The binary addition on $\mathbb{R}_+$ is defined by the formula

$$x \oplus y := \sum_{j=1}^{\infty} |x_{-j} - y_{-j}| 2^{j-1} + \sum_{j=1}^{\infty} |x_j - y_j| 2^{-j}, \quad x, y \in \mathbb{R}_+,$$

and plays a key role in the theory of Walsh–Fourier series (e.g., [16]). It is well-known that, for all $x \in \mathbb{R}_+$, $w_m(x) w_n(x) = w_{m \oplus n}(x)$, and, if $x \oplus y$ is a dyadic irrational, then

$$w_n(x \oplus y) = w_n(x) w_n(y). \tag{1.1}$$

Thus, for fixed y, equality (1.1) is valid for all $x \in \mathbb{R}_+$ except countably many of them. An interval $I \subset \mathbb{R}_+$ is a *dyadic interval of range* n if $I = I_k^{(n)} := [k2^{-n}, (k+1)2^{-n})$ for some $k \in \mathbb{Z}_+$. Let $\Delta := [0,1)$. It is easily seen that

$$I_k^{(n)} \cap I_l^{(n)} = \emptyset \quad \text{for} \quad k \neq l \quad \text{and} \quad \bigcup_{k=0}^{2^n-1} I_k^{(n)} = \Delta.$$

Moreover, it is clear that $w_l(x)$ is constant on $I_k^{(n)}$ for each $0 \le l \le 2^n - 1$ and $0 \le k \le 2^n - 1$. We shall use the notation

$$w_{l,k}^{(n)} := w_l(k2^{-n}) \quad \text{for } 0 \le l, k \le 2^n - 1.$$

Notice that

$$w_{0,0}^{(0)} = 1, \quad w_{0,0}^{(1)} = w_{1,0}^{(1)} = w_{0,1}^{(1)} = 1, \quad w_{1,1}^{(1)} = -1, \quad w_{l,k}^{(n)} = w_{k,l}^{(n)}, \tag{1.2}$$

$$\sum_{i=0}^{2^n-1} w_{i,l}^{(n)} w_{i,k}^{(n)} = \sum_{j=0}^{2^n-1} w_{l,j}^{(n)} w_{k,j}^{(n)} = 2^n \delta_{l,k}, \quad 0 \le l, k \le 2^n - 1. \tag{1.3}$$

Also, the following equalities hold:

$$w_{2l,k}^{(n+1)} = w_{2l+1,k}^{(n+1)} = w_{l,k}^{(n)}, \quad w_{2l,2^n+k}^{(n+1)} = -w_{2l+1,2^n+k}^{(n+1)} = w_{l,k}^{(n)}, \tag{1.4}$$

$$w_{l,2k}^{(n+1)} = w_{l,2k+1}^{(n+1)} = w_{l,k}^{(n)}, \quad w_{2^n+l,2k}^{(n+1)} = -w_{2^n+l,2k+1}^{(n+1)} = w_{l,k}^{(n)}. \tag{1.5}$$

To keep our notation simple, we write $N := 2^n$. A finite sum

$$D_N(x) := \sum_{j=0}^{N-1} w_j(x), \quad x \in \mathbb{R}_+,$$

is called the *Walsh-Dirichlet kernel* of order N. Paley's lemma [16, p. 7] states that

$$D_N(x) = \begin{cases} N, & x \in I_0^{(n)}, \\ 0, & x \in \Delta \setminus I_0^{(n)}. \end{cases} \tag{1.6}$$

It follows from (1.1) that the Nth partial sum of the Walsh-Fourier series of f is written as

$$S_N f(x) = \int_0^1 D_N(x \oplus t) f(t) dt, \quad x \in \Delta. \tag{1.7}$$

It is known that, for any $f \in L^1(\Delta)$,

$$\lim_{N\to\infty} S_N f = f \text{ a.e. on } \Delta \text{ and } \lim_{N\to\infty} \|f - S_N f\|_{L^1(\Delta)} = 0.$$

Moreover, the Walsh system $\{w_l \mid l \in \mathbb{Z}_+\}$ is a basis in $L^p(\Delta)$ for $1 < p < \infty$ and it is not a basis in $L^1(\Delta)$ (e.g., [9], [16]).

By analogy with [2], we introduce the following notations:

$$D_N^*(x) := \frac{1}{2} + \sum_{k=1}^{N-2} w_k(x) + \frac{1}{2} w_{N-1}(x), \quad x_{n,k} := \frac{k}{N},$$

$$\varphi_{n,k}(x) := \Phi_n(x \oplus x_{n,k}), \quad \psi_{n,k}(x) := \Psi_n(x \oplus x_{n,k}), \quad k = 0, 1, \ldots, N-1,$$

where

$$\Phi_n(x) := \int_0^{x_{n,1}} D_N^*(x \oplus t) dt, \tag{1.8}$$

$$\begin{aligned} \Psi_n(x) := & \int_0^{x_{n+1,1}} [D_{2N}^*(x \oplus t) - D_N^*(x \oplus t)] dt \\ & - \int_{x_{n+1,1}}^{x_{n,1}} [D_{2N}^*(x \oplus t) - D_N^*(x \oplus t)] dt. \end{aligned} \tag{1.9}$$

In Section 2 we prove that $\{\varphi_{n,k}\}_{k=0}^{N-1}$ and $\{\psi_{n,k}\}_{k=0}^{N-1}$ are bases for the spaces $V_n := \operatorname{span}\{1, w_1(x), \ldots, w_{N-1}(x)\}$, $W_n := \operatorname{span}\{w_N(x), w_{N+1}(x), \ldots, w_{2N-1}(x)\}$, respectively. Note that the orthogonal direct sum of V_n and W_n coincides with V_{n+1}, that is, $V_n \bigoplus W_n = V_{n+1}$. The spaces V_n and W_n will be called the *approximation spaces* and *wavelet spaces*, while the functions $\varphi_{n,k}$ and $\psi_{n,k}$ will be called the *scaling functions* and *wavelets*, respectively.

In Section 3 we give the algorithms for decomposition a function $f_{n+1} \in V_{n+1}$ into the direct sum:

$$f_{n+1} = f_n + g_n, \quad f_n \in V_n, \quad g_n \in W_n,$$

and for reconstruction f_{n+1} from f_n and g_n. Let $\psi^{(\gamma)}(x) = \mathscr{D}_{2\gamma}(x) - \mathscr{D}_\gamma(x)$, where $\mathscr{D}_\gamma$ is the generalized Walsh-Dirichlet kernel with a parameter γ. Suppose that $\gamma \in (0,1)$. In Section 4, using the Daubechies type "admissible condition", we prove that the system of functions

$$\psi_{jk}^{(\gamma)}(x) = 2^{j/2}\psi^{(\gamma)}(2^j x \oplus k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}_+,$$

is a Parseval frame for $L^2(\mathbb{R}_+)$. Moreover, it is shown that the subspaces

$$V_j(\gamma) = \{f \in L^2(\mathbb{R}_+) \,|\, \widehat{f}(\omega) = 0, \omega > 2^j\gamma\}, \quad j \in \mathbb{Z},$$

form a frame multiresolution analysis in $L^2(\mathbb{R}_+)$ with a scaling function $\varphi = \mathscr{D}_\gamma$. Some new examples of Parseval frames for $L^2(\mathbb{R}_+)$ are also given.

2. **Bases in Approximation and Wavelet Spaces**

For each polynomial $v \in V_n$ we have

$$v(x_{n,l}) = v(x) \quad \text{for all} \ \ x \in I_l^{(n)}, \quad 0 \le l \le N-1. \tag{2.1}$$

Moreover, we can use the discrete Walsh transform to recover v from the values $v(x_{n,l})$. Indeed, if

$$v(x) = \sum_{k=0}^{N-1} c_k w_k(x), \quad x \in \Delta, \tag{2.2}$$

than

$$c_k = \frac{1}{N}\sum_{l=0}^{N-1} w_{l,k}^{(n)} v(x_{n,l}), \quad 0 \le k \le N-1; \tag{2.3}$$

see, e.g., [10], [16], [17], where the corresponding fast algorithms are given.

As a consequence of (1.1) we observe that

$$\int_0^{x_{n,1}} w_k(x \oplus t)dt = \int_0^{1/N} w_k(x)w_k(t)dt = N^{-1}w_k(x), \quad 0 \le k \le N-1.$$

Hence, since $w_k(t) = 1$ for all $t \in [0, 1/N)$, we obtain from (1.8) that

$$\Phi_n(x) = N^{-1}D_N^*(x), \quad x \in \Delta. \tag{2.4}$$

By (2.4) and the definition of $\varphi_{n,k}$ it is easy to see that

$$N\,\varphi_{n,k}(x) = \frac{1}{2} + \sum_{j=1}^{N-2} w_{k,j}^{(n)} w_j(x) + \frac{1}{2} w_{k,N-1}^{(n)} w_{N-1}(x), \quad 0 \le k \le N-1. \tag{2.5}$$

Furthermore, we have by (1.3) that

$$\sum_{k=0}^{N-1} w_{l,k}^{(n)}\,\varphi_{n,k}(x) = w_l(x), \quad 1 \le l \le N-2, \tag{2.6}$$

$$\sum_{k=0}^{N-1} \varphi_{n,k}(x) = \frac{1}{2}, \qquad \sum_{k=0}^{N-1} w_{N-1,k}^{(n)}\,\varphi_{n,k}(x) = \frac{1}{2} w_{N-1}(x). \tag{2.7}$$

For each $l \in \{0, 1, \ldots, N-1\}$ with binary expansion

$$l = \sum_{j=0}^{n-1} \nu_j 2^j, \quad \nu_j \in \{0, 1\},$$

we let $\sigma(l) = 1$, if among the ν_j there are odd units, and let $\sigma(l) = 0$ otherwise.

Theorem 2.1. *Let $v \in V_n$. Assume that $\alpha_{n,k} = \alpha_{n,k}(v)$, $k = 0, 1, \dots, N-1$, are defined by*

$$\alpha_{n,k} = \begin{cases} v(x_{n,k}) + 2N^{-1} \sum_{l=0}^{N-1} (1-\sigma(l)) v(x_{n,l}), & \text{if } \sigma(k) = 0, \\ v(x_{n,k}) + 2N^{-1} \sum_{l=0}^{N-1} \sigma(l) v(x_{n,l}), & \text{if } \sigma(k) = 1. \end{cases} \tag{2.8}$$

Then

$$v(x) = \sum_{k=0}^{N-1} \alpha_{n,k} \varphi_{n,k}(x), \quad x \in \Delta. \tag{2.9}$$

Proof. According to (2.2), (2.6) and (2.7) we have

$$v(x) = 2c_0 \sum_{k=0}^{N-1} \varphi_{n,k}(x) + \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} c_l w_{l,k}^{(n)} \varphi_{n,k}(x) + 2c_{N-1} \sum_{k=0}^{N-1} w_{N-1,k}^{(n)} \varphi_{n,k}(x).$$

Therefore, equality (2.9) is true with the coefficients

$$\alpha_{n,k} = 2c_0 + 2c_{N-1} w_{N-1,k}^{(n)} + \sum_{l=0}^{N-1} c_l w_{l,k}^{(n)}, \quad k = 0, 1, \dots, N-1. \tag{2.10}$$

But from (2.2) and (2.3) it follows that

$$\sum_{l=0}^{N-1} c_l w_{l,k}^{(n)} = \sum_{l=0}^{N-1} c_l w_l(x_{n,k}) = v(x_{n,k}), \qquad w_{N-1,k}^{(n)} = (-1)^{\sigma(k)},$$

$$c_0 = \frac{1}{N} \sum_{l=0}^{N-1} v(x_{n,l}), \quad c_{N-1} = \frac{1}{N} \sum_{l=0}^{N-1} (-1)^{\sigma(l)} v(x_{n,l}).$$

Combining these equalities with (2.10), we obtain (2.8). □

Remark 2.2. If coefficients c_k are known, then $\alpha_{n,k}$ can be computed by (2.10).

Proposition 2.3. *The following equalities hold*

$$D_N(x) - \frac{1}{2} = N[\varphi_{n+1,0}(x) + \varphi_{n+1,1}(x)], \tag{2.11}$$

$$\Phi_n(x) = \varphi_{n+1,0}(x) + \varphi_{n+1,1}(x) - \frac{1}{2N} w_{N-1}(x), \tag{2.12}$$

$$\Psi_n(x) = \varphi_{n+1,0}(x) - \varphi_{n+1,1}(x). \tag{2.13}$$

Proof. We see from (1.6) that $D_N(0) = D_N(1/2N) = N$ and $D_N(k/2N) = 0$ for $k = 2, 3, \dots, 2N-1$. Thus applying (2.9) with n replaced by $n+1$ to the polynomial $v = D_N$ we arrive at

$$\begin{aligned} D_N(x) &= (N+1)(\varphi_{n+1,0}(x) + \varphi_{n+1,1}(x)) + \sum_{k=2}^{2N-1} \varphi_{n+1,k}(x) \\ &= N(\varphi_{n+1,0}(x) + \varphi_{n+1,1}(x)) + \sum_{k=0}^{2N-1} \varphi_{n+1,k}(x) \end{aligned}$$

and by (2.7) we then obtain (2.11). Now, since

$$D_N(x) - \frac{1}{2} = D_N^*(x) + \frac{1}{2} w_{N-1}(x),$$

we see that (2.12) follows from (2.4) and (2.11) . Observing that

$$w_{1,k}^{(n+1)} = w_1(k/2N) = \begin{cases} 1, & 0 \le k \le N-1, \\ -1, & N \le k \le 2N-1, \end{cases}$$

by (2.5) we get

$$\varphi_{n+1,1}(x) = \frac{1}{2N} \left(2D_N(x) - D_{2N}(x) - \frac{1}{2} - \frac{1}{2} w_{N-1}(x) \right). \qquad (2.14)$$

Further, from

$$\int_0^{1/2N} w_k(x \oplus t)dt = \frac{1}{2N} w_k(x), \quad 0 \le k \le 2N-1,$$

$$\int_{1/2N}^{1/N} w_k(x \oplus t)dt = \begin{cases} w_k(x)/(2N), & 0 \le k \le N-1, \\ -w_k(x)/(2N), & N \le k \le 2N-1, \end{cases}$$

we deduce that

$$\int_0^{x_{n+1,1}} [D_{2N}^*(x \oplus t) - D_N^*(x \oplus t)]dt$$
$$= \frac{1}{2N} \left(\frac{1}{2} w_{N-1}(x) + \sum_{k=N}^{2N-1} w_k(x) + \frac{1}{2} w_{2N-1}(x) \right),$$
$$\int_{x_{n+1,1}}^{x_{n,1}} [D_{2N}^*(x \oplus t) - D_N^*(x \oplus t)]dt$$
$$= \frac{1}{2N} \left(\frac{1}{2} w_{N-1}(x) - \sum_{k=N}^{2N-1} w_k(x) - \frac{1}{2} w_{2N-1}(x) \right).$$

Therefore,

$$\begin{aligned} \Psi(x) &= \frac{1}{N} \left(\sum_{k=N}^{2N-1} w_k(x) + \frac{1}{2} w_{2N-1}(x) \right) \\ &= \frac{1}{N} \left(D_{2N}(x) - D_N(x) + \frac{1}{2} w_{2N-1}(x) \right), \end{aligned}$$

which by (2.14) yields

$$\Psi(x) + 2\varphi_{n+1,1}(x) = \frac{1}{N} \left(D_N(x) - \frac{1}{2} \right).$$

From this and (2.11) we obtain (2.13). □

Note that (2.12) and (2.13) with x replaced by $x \oplus x_{n,k}$ give the following equalities

$$\varphi_{n,k}(x) = \varphi_{n+1,2k}(x) + \varphi_{n+1,2k+1}(x) - \frac{(-1)^{\sigma(k)}}{2N} w_{N-1}(x), \tag{2.15}$$

$$\psi_{n,k}(x) = \varphi_{n+1,2k}(x) - \varphi_{n+1,2k+1}(x). \tag{2.16}$$

Hence,

$$\varphi_{n+1,2k}(x) = \frac{1}{2}\varphi_{n,k}(x) + \frac{1}{2}\psi_{n,k}(x) + \frac{(-1)^{\sigma(k)}}{4N} w_{N-1}(x), \tag{2.17}$$

$$\varphi_{n+1,2k+1}(x) = \frac{1}{2}\varphi_{n,k}(x) - \frac{1}{2}\psi_{n,k}(x) - \frac{(-1)^{\sigma(k)}}{4N} w_{N-1}(x). \tag{2.18}$$

Proposition 2.4. *For any n and $N = 2^n$,*

$$w_{N-1}(x) = \sum_{k=0}^{2N-1} \gamma_{n+1,k}\, \varphi_{n+1,k}(x), \tag{2.19}$$

where $\gamma_{n+1,k}$ are given by

$$\gamma_{2,0} = \gamma_{2,1} = 1, \quad \gamma_{2,2} = \gamma_{2,3} = -1, \tag{2.20}$$

$$\gamma_{n+1,k} = \gamma_{n,k}, \quad \gamma_{n+1,N+k} = -\gamma_{n,k}, \quad n \ge 2, \quad 0 \le k \le N-1. \tag{2.21}$$

Proof. Using Theorem 2.1, from (2.19) we get

$$\gamma_{n+1,k} = \begin{cases} w_{N-1}(k/2N) + N^{-1} \sum_{l=0}^{2N-1} (1-\sigma(l)) w_{N-1}(l/2N), & \sigma(k) = 0, \\ w_{N-1}(k/2N) + N^{-1} \sum_{l=0}^{N-1} \sigma(l) w_{N-1}(l/2N), & \sigma(k) = 1, \end{cases}$$

where $w_{N-1}(k/2N) = (-1)^{\sigma(k)}$ if k is even and $w_{N-1}(k/2N) = (-1)^{\sigma(k)+1}$ if k is odd. This immediately gives (2.20) and (2.21). □

Note that Proposition 2.4 can be deduced also from (1.4), (1.5) and (2.7).

We see that formulas (2.15), (2.16) and (2.19) express $\varphi_{n,k}$ and $\psi_{n,k}$, the wavelets and scaling functions *at level n*, in terms of scaling functions at level $n+1$. Conversely, substituting the expression

$$w_{N-1}(x) = 2\sum_{j=0}^{N-1} (-1)^{\sigma(j)} \varphi_{n,j}(x) \tag{2.22}$$

in (2.17) and (2.18), we obtain the representations of scaling functions $\varphi_{n+1,l}$, $0 \le l \le 2N-1$, by the wavelets and scaling functions at level n. The following theorem shows that, for each $n \in \mathbb{Z}_+$, the functions $\psi_{n,k}$ given in Section 1, form the basis for W_n.

Theorem 2.5. *Let $v \in W_n$. Then*

$$v(x) = \sum_{k=0}^{N-1} \beta_{n,k} \psi_{n,k}(x), \quad x \in \Delta, \tag{2.23}$$

where, with the notations as in (2.8),

$$\beta_{n,k} = \beta_{n,k}(v) = \alpha_{n+1,2k} = -\alpha_{n+1,2k+1}, \quad 0 \le k \le N-1. \tag{2.24}$$

Proof. Since $v \in W_n$, we have

$$v(x) = \sum_{k=0}^{2N-1} c_k w_k(x)$$

with $c_0 = c_1 = \cdots = c_{N-1} = 0$. Thus, from (1.5) and (2.10) for $k = 0, 1, \ldots, N-1$ it follows that

$$\begin{aligned}
\alpha_{n+1,2k} &= 2c_{2N-1} w^{(n+1)}_{2N-1,2k} + \sum_{l=0}^{N-2} c_{N+l} w^{(n+1)}_{N+l,2k} \\
&= -2c_{2N-1} w^{(n+1)}_{2N-1,2k+1} - \sum_{l=0}^{N-2} c_{N+l} w^{(n+1)}_{N+l,2k+1} \\
&= -\alpha_{n+1,2k+1}.
\end{aligned}$$

Hence, $\beta_{n,k}$ can be defined by (2.24). Then, since $v \in W_n \subset V_{n+1}$, by Theorem 2.1 and (2.16) we get

$$\begin{aligned}
v(x) &= \sum_{k=0}^{N-1} \alpha_{n+1,2k} \varphi_{n+1,2k}(x) + \sum_{k=0}^{N-1} \alpha_{n+1,2k+1} \varphi_{n+1,2k+1}(x) \\
&= \sum_{k=0}^{N-1} \beta_{n,k} \psi_{n,k}(x).
\end{aligned}$$

□

3. **Algorithms**

For functions $f_n \in V_n$ and $g_n \in W_n$ we write

$$f_n(x) = \sum_{k=0}^{N-1} C_{n,k} \varphi_{n,k}(x), \quad g_n(x) = \sum_{k=0}^{N-1} D_{n,k} \psi_{n,k}(x), \tag{3.1}$$

where the coefficient sequences

$$\mathbf{C}_n = \{C_{n,k}\}, \quad \mathbf{D}_n = \{C_{n,k}\} \tag{3.2}$$

uniquely determine f_n and g_n, respectively. In this section we describe the algorithms, in terms of these coefficient sequences, for decomposing $f_{n+1} \in V_{n+1}$ as the orthogonal sum of $f_n \in V_n$ and $g_n \in W_n$, and for reconstructing f_{n+1} from f_n and g_n (cf. [2, § 5]).

According to (2.17), (2.18) and (2.22) we have

$$\varphi_{n+1,l}(x) = \frac{1}{2}\varphi_{n,[l/2]}(x) + \frac{(-1)^l}{2}\psi_{n,[l/2]}(x) + \frac{(-1)^{l+\sigma(l)}}{4N}w_{N-1}(x),$$

where

$$w_{N-1}(x) = 2\sum_{k=0}^{N-1}(-1)^{\sigma(k)}\varphi_{n,k}(x).$$

Hence,

$$\varphi_{n+1,l}(x) = \sum_k A_{l,k}^{(n)}\varphi_{n,k}(x) + B_{l,k}^{(n)}\psi_{n,k}(x), \tag{3.3}$$

where

$$A_{l,k}^{(n)} = \begin{cases} \left[1/2 + (-1)^{l+\sigma(l)}/(2N)\right]\left[1/2 + (-1)^{\sigma([l/2])}/(2N)\right], & k = [l/2], \\ (2N)^{-1}(-1)^{l+\sigma(l)+\sigma(k)}, & k \neq [l/2], \end{cases}$$

$$B_{l,k}^{(n)} = \begin{cases} (-1)^l/2, & k = [l/2], \\ 0, & k \neq [l/2]. \end{cases}$$

Thus, in view of (3.1) and (3.3),

$$\begin{aligned} &\sum_k C_{n,k}\varphi_{n,k}(x) + \sum_k D_{n,k}\psi_{n,k}(x) \\ &\quad = \sum_l C_{n+1,l}\left\{\sum_k A_{l,k}^{(n)}\varphi_{n,k}(x) + B_{l,k}^{(n)}\psi_{n,k}(x)\right\} \\ &\quad = \sum_k\left\{\sum_l C_{n+1,l}A_{l,k}^{(n)}\right\}\varphi_{n,k}(x) + \sum_k\left\{\sum_l C_{n+1,l}B_{l,k}^{(n)}\right\}\psi_{n,k}(x). \end{aligned}$$

This implies that

$$C_{n,k} = \sum_l A_{l,k}^{(n)}C_{n+1,l}, \quad D_{n,k} = \sum_l B_{l,k}^{(n)}C_{n+1,l}. \tag{3.4}$$

Further, using (2.15), (2.16) and (2.19), we obtain

$$\varphi_{n,l}(x) = \sum_k P_{l,k}^{(n)}\varphi_{n+1,k}(x) + Q_{l,k}^{(n)}\varphi_{n+1,k}(x),$$

where

$$P_{l,k}^{(n)} = \begin{cases} 1 + (-1)^{\sigma(l)+1}\gamma_{n+1,k}/(2N), & \text{if } k = 2l \text{ or } k = 2l+1, \\ \gamma_{n+1,k}, & \text{otherwise,} \end{cases}$$

$$Q_{l,k}^{(n)} = \begin{cases} 1, & \text{if } k = 2l, \\ -1, & \text{if } k = 2l+1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\sum_k C_{n+1,k}\,\varphi_{n+1,k}(x)$$
$$= \sum_l C_{n,l}\left\{\sum_k P^{(n)}_{l,k}\,\varphi_{n+1,k}(x)\right\} + \sum_l D_{n,l}\left\{\sum_k Q^{(n)}_{l,k}\,\varphi_{n+1,k}(x)\right\}$$
$$= \sum_k \left\{\sum_l P^{(n)}_{l,k} C_{n,l} + Q^{(n)}_{l,k} D_{n,l}\right\}\varphi_{n+1,k}(x)$$

and so

$$C_{n+1,k} = \sum_l P^{(n)}_{l,k} C_{n,l} + Q^{(n)}_{l,k} D_{n,l}\,. \tag{3.5}$$

We remark that the decomposition and reconstruction algorithms based on formulas (3.4) and (3.5) have more simply structure than the similar algorithms constructed in [2] for the case of trigonometric wavelets.

4. **Examples of Frames for** $L^2(\mathbb{R}_+)$

Let $\mathscr{H}$ be a Hilbert space and let M be a countable set. We recall that a family $\{g_m |\ m \in M\}$ is a *frame* for $\mathscr{H}$ if there exist positive constants A and B such that, for every $f \in \mathscr{H}$,

$$A\|f\|^2 \le \sum_{m\in M} |\langle f, g_m\rangle|^2 \le B\|f\|^2.$$

The constants A and B are known respectively as lower and upper frame bounds. A frame is called a *tight frame* if the lower and upper frame bounds are equal; $A = B$. A frame is a *Parseval frame* if $A = B = 1$. The following two propositions are well-known (e.g., [11, p. 142], [15]):

Proposition 4.1. *A sequence* $\{g_m\}$ *is a Parseval frame for a Hilbert space* $\mathscr{H}$ *if and only if the following formula holds for every* $f \in \mathscr{H}$*:*

$$f = \sum_{m\in M} \langle f, g_m\rangle g_m.$$

Proposition 4.2. *Let* $\{g_m\}$ *be a frame for* $\mathscr{H}$ *and let* $\mathscr{P} : \mathscr{H} \to \mathscr{H}$ *be an orthogonal projection. Then* $\{\mathscr{P} g_m\}$ *is a frame for* $\mathscr{P}(\mathscr{H})$ *with the same frame bounds. In particular, if* $\{g_m\}$ *is an orthonormal basis for* $\mathscr{H}$*, then* $\{\mathscr{P} g_m\}$ *is a Parseval frame for* $\mathscr{P}(\mathscr{H})$.

A function $f : \mathbb{R}_+ \to \mathbb{C}$ is said to be *W-continuous at a point* $x \in \mathbb{R}_+$, if

$$\sup_{0\le h<1/2^n} |f(x \oplus h) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

A function f is *W-continuous* if it is W-continuous at each point of $\mathbb{R}_+$. The Walsh functions $\{w_l\}$ are W-continuous (e.g., [16], § 1.3).

For $x, \omega \in \mathbb{R}_+$ we set

$$\chi(x,\omega) = (-1)^{\sigma(x,\omega)}, \quad \sigma(x,\omega) = \sum_{j=1}^{\infty} x_j \omega_{-j} + x_{-j}\omega_j.$$

The *Walsh-Fourier transform* of a function $f \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ is defined by

$$\widehat{f}(\omega) = \int_{\mathbb{R}_+} f(x)\chi(x,\omega)dx, \quad \omega \in \mathbb{R}_+,$$

and admits a standard extension to the whole space $L^2(\mathbb{R}_+)$. Denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and the norm in $L^2(\mathbb{R}_+)$, respectively. In the next proposition we list several well-known properties of the Walsh-Fourier transform.

Proposition 4.3. *The following properties take place:*

(a) *If $f \in L^1(\mathbb{R}_+)$, then $\widehat{f}$ is a W-continuous function and $\widehat{f}(\omega) \to 0$ as $\omega \to \infty$.*

(b) *Suppose that both f and $\widehat{f}$ belong to $L^1(\mathbb{R}_+)$. If f is W-continuous at a point x, then the following inversion formula*

$$f(x) = \int_{\mathbb{R}_+} \widehat{f}(\omega)\chi(x,\omega)d\omega$$

holds.

(c) *If $f \in L^2(\mathbb{R}_+)$, then $\widehat{f} \in L^2(\mathbb{R}_+)$ and $\|\widehat{f}\| = \|f\|$.*

For any function $f \in L^2(\mathbb{R}_+)$ we set

$$f_{jk}(x) := 2^{j/2} f(2^j x \oplus k), \quad j \in \mathbb{Z}, \ \ k \in \mathbb{Z}_+, \ \ x \in \mathbb{R}_+.$$

It is easily seen that

$$\widehat{f}_{jk}(\omega) = 2^{-j/2} w_k(2^{-j}\omega)\widehat{f}(2^{-j}\omega)$$

and, by Plancherel's theorem, for all $g \in L^2(\mathbb{R}_+)$,

$$\langle g, f_{jk} \rangle = \langle \widehat{g}, \widehat{f}_{jk} \rangle = 2^{-j/2} \int_{\mathbb{R}_+} \widehat{g}(\omega)\overline{\widehat{f}(2^{-j}\omega)} w_k(2^{-j}\omega)d\omega. \tag{4.1}$$

The *Haar wavelet* on $\mathbb{R}_+$ is defined by

$$\psi_H(x) := \begin{cases} 1, & x \in [0, 1/2), \\ -1, & x \in [1/2, 1), \\ 0, & x \in [1, +\infty). \end{cases}$$

It is well-known that if $f = \psi_H$, then $\{f_{jk} \,|\, j \in \mathbb{Z}, \ k \in \mathbb{Z}_+\}$ is an orthonormal bases for $L^2(\mathbb{R}_+)$. In general, we say that a function ψ is *wavelet in* $L^2(\mathbb{R}_+)$, if the following condition is satisfied:

$$0 < c_\psi := \int_{\mathbb{R}_+} |\widehat{\psi}(\omega)|^2 \frac{d\omega}{\omega} < +\infty. \tag{4.2}$$

The *continuous wavelet transform* of a function $f \in L^2(\mathbb{R}_+)$ with analyzing wavelet ψ is defined by the integral:

$$(W_\psi f)(a,b) = a^{-1/2} \int_{\mathbb{R}_+} f(x)\overline{\psi((x\oplus b)/a)}\,dx, \quad a>0,\ b\in\mathbb{R}_+.$$

This transform was studied in [6], where an analog of the Grossmann and Morlet theorem is proved. Note that some discrete constructions of this type give frames for $L^2(\mathbb{R}_+)$ (cf. [14, § 5.2]).

Now, for $m\in\mathbb{Z}_+$, $l\in\mathbb{N}$ we set

$$a_{ml}(x) = \begin{cases} 2^{-m/2}w_l(2^{-m}x), & x\in[0,2^m),\\ 0, & x\in[2^m,+\infty). \end{cases} \tag{4.3}$$

We see that $a_{0,1}$ coincides with the Haar wavelet ψ_H. Besides, the following properties take place:

1. For each fixed $m \in \mathbb{Z}_+$ the system $\{a_{ml}\,|\,l\in\mathbb{N}\}$ satisfy the orthogonality condition on $[0,2^m)$, that is,
$$\int_0^{2^m} a_{ml}(x)a_{mk}(x)dx = \delta_{l,k}, \quad l,k\in\mathbb{N}.$$
2. The Walsh-Fourier transform of a_{ml} can be written as
$$\widehat{a}_{ml}(\omega) = \begin{cases} 2^{m/2}, & \omega\in[l2^{-m},(l+1)2^{-m}),\\ 0, & \omega\notin[l2^{-m},(l+1)2^{-m}). \end{cases}$$
3. Each function a_{ml} is a wavelet in $L^2(\mathbb{R}_+)$ (if $\psi = a_{ml}$, then (4.2) is satisfied with $c_\psi = 2^m\log(1+1/l)$).
4. There exist orthogonal wavelets in $L^2(\mathbb{R}_+)$ which are finite linear combinations of functions a_{ml}.

The proofs of properties 1-3 are straightforward while the last property follows from examples of dyadic wavelets on $\mathbb{R}_+$ given in [5].

For $\alpha>0$, $m\in\mathbb{Z}_+$, $l\in\mathbb{N}$ let $g_{ml}^{(\alpha)}$ be a function in $L^2(\mathbb{R}_+)$ such that

$$\widehat{g}_{ml}^{(\alpha)}(\omega) = \begin{cases} 1, & \omega\in[l\alpha^{-m},(l+1)\alpha^{-m}),\\ 0, & \omega\notin[l\alpha^{-m},(l+1)\alpha^{-m}). \end{cases}$$

One can see that each $g_{ml}^{(\alpha)}$ is wavelet in $L^2(\mathbb{R}_+)$. Returning to the general case, we define

$$D_\psi(\omega) := \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^{-j}\omega)|^2, \quad M_{l,\psi} := \sup_{\omega\in\mathbb{R}_+}\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^{-j}\omega)||\widehat{\psi}(2^{-j}\omega\oplus l)|. \tag{4.4}$$

We note that $D_\psi(\omega) = D_\psi(2\omega)$ for all $\omega\in\mathbb{R}_+$, and that sup in (4.4) can be taken over $1\le\omega<2$. The following theorem similar to the well-known Daubechies result (e.g., [14, § 5.3]).

Theorem 4.4. *Let $\psi \in L^2(\mathbb{R}_+)$ be such that*

$$A_\psi := \operatorname*{ess\,inf}_{\omega\in\mathbb{R}_+} D_\psi(\omega) - \sum_{l\in\mathbb{N}} M_{l,\psi} > 0$$

and

$$B_\psi := \operatorname*{ess\,sup}_{\omega\in\mathbb{R}_+} D_\psi(\omega) + \sum_{l\in\mathbb{N}} M_{l,\psi} < \infty.$$

Then $\{\psi_{jk}\}$ is a frame with frame bounds A_ψ and B_ψ.

Proof. For $j \in \mathbb{Z}$, $l \in \mathbb{Z}_+$ we have

$$\begin{aligned}\int_{l2^j}^{(l+1)2^j} w_k(2^{-j}\omega)d\omega &= \int_0^{2^j} w_k(2^{-j}(\omega \oplus l2^j)d\omega \\ &= \int_0^{2^j} w_k(2^{-j}\omega)d\omega.\end{aligned}$$

Let $f \in L^2(\mathbb{R}_+)$. Then, according to (4.1),

$$\langle f, \psi_{jk}\rangle = 2^{-j/2}\int_0^{2^j} \left(\sum_{l\in\mathbb{Z}_+} F_{jl}(\omega)\right) w_k(2^{-j}\omega)d\omega, \tag{4.5}$$

where $F_{jl}(\omega) := \widehat{f}(\omega \oplus l2^j)\overline{\widehat{\psi}(2^{-j}\omega \oplus l)}$. Now, for each j let F_j be the function defined by

$$F_j(\omega) := \sum_{l\in\mathbb{Z}_+} F_{jl}(\omega).$$

This function is periodic: $F_j(\omega \oplus 2^j) = F_j(\omega)$ for all $\omega \in \mathbb{R}_+$, and one sees easily that F_j can be expanded to the Walsh series:

$$F_j(\omega) = \sum_k c_k(F_j) w_k(2^{-j}\omega), \quad \omega \in [0, 2^j),$$

where

$$c_k(F_j) = 2^{-j}\int_0^{2^j} F_j(\omega)w_k(2^{-j}\omega)d\omega.$$

By Parseval's formula,

$$\sum_k |c_k(F_j)|^2 = 2^{-j}\int_0^{2^j} |F_j(\omega)|^2 d\omega.$$

Therefore, in view of (4.5),

$$\sum_{j,k} |\langle f, \psi_{jk}\rangle|^2 = \sum_{j,k} 2^{-j}\left|\int_0^{2^j} F_j(\omega)w_k(2^{-j}\omega)d\omega\right|^2$$

$$= \sum_j 2^j \sum_k |c_k(F_j)|^2$$
$$= \sum_j \int_0^{2^j} |F_j(\omega)|^2 d\omega.$$

Also, for any $j \in \mathbb{Z}$ we have

$$\begin{aligned} \int_0^{2^j} F_j(\omega)\overline{F_j(\omega)}d\omega &= \int_0^{2^j} \left(\sum_{l\in\mathbb{Z}_+} \widehat{f}(\omega \oplus l2^j)\overline{\widehat{\psi}(2^{-j}\omega \oplus l)}\overline{F_j(\omega)} \right) d\omega \\ &= \sum_{l\in\mathbb{Z}_+} \int_{l2^j}^{(l+1)2^j} \widehat{f}(\omega)\overline{\widehat{\psi}(2^{-j}\omega)}\overline{F_j(\omega)}d\omega \\ &= \int_{\mathbb{R}_+} \widehat{f}(\omega)\overline{\widehat{\psi}(2^{-j}\omega)}\overline{F_j(\omega)}d\omega \end{aligned}$$

and so

$$\begin{aligned} \sum_{j,k} |\langle f, \psi_{jk}\rangle|^2 &= \sum_j \int_{\mathbb{R}_+} \widehat{f}(\omega)\overline{\widehat{\psi}(2^{-j}\omega)}\overline{F_j(\omega)}d\omega \\ &= \int_{\mathbb{R}_+} |\widehat{f}(\omega)|^2 \sum_j |\widehat{\psi}(2^{-j}\omega)|^2 d\omega + R(f), \end{aligned}$$

where, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |R(f)| &= \left| \sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \int_{\mathbb{R}_+} \widehat{f}(\omega)\overline{\widehat{f}(\omega \oplus l2^j)\widehat{\psi}(2^{-j}\omega)}\psi(2^{-j}\omega \oplus l)d\omega \right| \\ &\le \sum_{l\in\mathbb{N}} \int_{\mathbb{R}_+} \left(|\widehat{f}(\omega)|^2 \sum_j |\widehat{\psi}(2^{-j}\omega)||\widehat{\psi}(2^{-j}\omega \oplus l)| \right) d\omega. \end{aligned}$$

Thus, for every $f \in L^2(\mathbb{R}_+)$,

$$A_\psi \|f\|^2 \le \sum_{j,k} |\langle f, \psi_{jk}\rangle|^2 \le B_\psi \|f\|^2,$$

where A_ψ and B_ψ are defined in the theorem. □

EXAMPLE 1. Let $\psi = g_{ms}^{(\alpha)}$, where $\alpha \ge 1, m \in \mathbb{Z}_+, s \in \mathbb{N}$. Then for any $l \in \mathbb{N}$ the supports of $\widehat{\psi}(2^{-j}\omega)$ and $\widehat{\psi}(2^{-j}\omega \oplus l)$ are disjoint. Since

$$\operatorname*{ess\,inf}_{1\le\omega<2} D_\psi(\omega) = \operatorname*{ess\,sup}_{1\le\omega<2} D_\psi(\omega) = 1,$$

we see that $A_\psi = B_\psi = 1$. Therefore, $\{\psi_{jk}\}$ is a Parseval frame for $L^2(\mathbb{R}_+)$. By setting $\alpha = 2$, we find also that each function a_{ms} generates a Parseval frame for $L^2(\mathbb{R}_+)$.

In the sequel, $\mathbf{1}_E$ stands for the characteristic function of a subset E of $\mathbb{R}_+$.

Example 4.5. Let us assume that

$$\psi(x) = 2^{-1/2}(a_{14}(x) + \nu a_{11}(x)),$$

where ν is a positive parameter. Then, for every $\omega \in \mathbb{R}_+$,

$$\widehat{\psi}(\omega) = \mathbf{1}_{[2,5/2)}(\omega) + \nu \mathbf{1}_{[1/2,1)}(\omega), \quad |\widehat{\psi}(\omega)|^2 = \mathbf{1}_{[2,5/2)}(\omega) + \nu^2 \mathbf{1}_{[1/2,1)}(\omega),$$

and

$$\widehat{\psi}(2^{-j}\omega)\widehat{\psi}(2^{-j}\omega \oplus l) = 0 \quad \text{for all} \quad j \in \mathbb{Z}_+,\ l \in \mathbb{N}.$$

Further, if $1 \le \omega < 5/4$ then $\widehat{\psi}(\omega/2) = \nu$, $\widehat{\psi}(2\omega) = 1$, and if $5/4 \le \omega < 2$ then $\widehat{\psi}(\omega/2) = \nu$. Besides, $\widehat{\psi}(2^{-j}\omega) = 0$ for $j \neq -1$ and $j \neq 1$. From these equalities we deduce that

$$A_\psi = \nu^2, \quad B_\psi = 1 + \nu^2, \quad B_\psi/A_\psi = 1 + \frac{1}{\nu^2}.$$

Thus, $\{\psi_{jk}\}$ tends to the tight frame when $\nu \to \infty$.

Let $\mathscr{E}_n$ be the space of functions which are constant on all dyadic intervals of range n. It is clear from the definition that, for every $f \in \mathscr{E}_n$,

$$f(x) = \sum_{k=0}^{\infty} f(k2^{-n})\mathbf{1}_{[k2^{-n},(k+1)2^{-n})}(x), \quad x \in \mathbb{R}_+. \tag{4.6}$$

The following two properties (see [10, § 6.2], [16, p. 461]) are known:

- if $f \in L^1(\mathbb{R}_+) \cap \mathscr{E}_n$, then $\operatorname{supp}\widehat{f} \subset [0, 2^n]$;
- if $f \in L^1(\mathbb{R}_+)$ and $\operatorname{supp} f \subset [0, 2^n]$, then $\widehat{f} \in \mathscr{E}_n$.

The generalized Walsh-Dirichlet kernel $\mathscr{D}_t$ with $t > 0$ is defined by

$$\mathscr{D}_t(x) := \int_0^t \chi(x,\omega)\, d\omega, \quad x \in \mathbb{R}_+.$$

It is known also that

$$\mathscr{D}_{2^n} = 2^n \mathbf{1}_{[0,2^{-n})} \quad \text{for all } n \in \mathbb{Z} \tag{4.7}$$

and

$$\widehat{\mathscr{D}}_t = \mathbf{1}_{[0,t)} \quad \text{for all } t > 0. \tag{4.8}$$

Example 3. Let $\psi = \mathscr{D}_{2\gamma} - \mathscr{D}_\gamma$, where $0 < \gamma \le 1$. Putting $\alpha = 1/\gamma$, from (4.8) we have $\widehat{\psi} = \mathbf{1}_{[1/\alpha,2/\alpha)}$; that is, $\psi = g_{11}^{(\alpha)}$ with $\alpha \ge 1$. According to Example 1, $\{\psi_{jk}\}$ is a Parseval frame for $L^2(\mathbb{R}_+)$.

Observe that, for any $t > 0$, the subspaces

$$V_j(t) := \{f \in L^2(\mathbb{R}_+) |\ \widehat{f}(\omega) = 0, \omega > 2^j t\}, \quad j \in \mathbb{Z}, \tag{4.9}$$

satisfy the following:

$$V_j(t) \subset V_{j+1}(t), \quad \bigcap_j V_j(t) = \{0\}, \quad \overline{\bigcup_j V_j(t)} = L^2(\mathbb{R}_+).$$

Theorem 4.6. *Let φ be the generalized Walsh-Dirichlet kernel $\mathscr{D}_t$ with $0 < t \leq 1$. Then*

$$\varphi(x) = \sum_{k \in \mathbb{Z}_+} \varphi(k/2)\,\varphi(2x \oplus k), \quad x \in \mathbb{R}_+. \tag{4.10}$$

Moreover, for each $j \in \mathbb{Z}$ the system $\{\varphi_{jk} \mid k \in \mathbb{Z}_+\}$ is a Parseval frame for $V_j(t)$.

Proof. For $t = 1$ we have $\varphi = \mathbf{1}_{[0,1)}$ and the subspaces $V_j(t)$ form the Haar multiresolution analysis in $L^2(\mathbb{R}_+)$. In this case we can write equation (4.10) as follows:

$$\varphi(x) = \varphi(2x) + \varphi(2x \oplus 1).$$

Now, let $0 < t < 1$ and assume that $E = [0, t)$. Then the linear mapping

$$\mathscr{P} : L^2[0,1] \to L^2[0,1], \quad \mathscr{P} f = f \cdot \mathbf{1}_E,$$

is an orthogonal projection. In fact, let $\mathscr{L}_0(E)$ be the closure of the linear span of $\{w_k \cdot \mathbf{1}_E \mid k \in \mathbb{Z}_+\}$ in $L^2[0,1]$. If $f \in L^2[0,1]$ and $g \in \mathscr{L}_0(E)$, then

$$\begin{aligned}
\langle f, g \rangle &= \int_0^1 f(t)\overline{g(t)}dt \\
&= \int_E f(t)\overline{g(t)}dt \\
&= \int_E \mathscr{P} f(t)\overline{g(t)}dt \\
&= \langle \mathscr{P} f, g \rangle.
\end{aligned}$$

Hence,

$$\langle f - \mathscr{P} f, g \rangle = 0 \quad \text{for all } g \in \mathscr{L}_0(E).$$

Since $\{w_k \mid k \in \mathbb{Z}_+\}$ is an orthonormal basis for $f \in L^2[0,1]$, by Proposition 4.2 we obtain that $\{w_k \cdot \mathbf{1}_E \mid k \in \mathbb{Z}_+\}$ is a Parseval frame for $\mathscr{L}_0(E)$. Recall that $\varphi_{0,k}(\cdot) = \varphi(\cdot \oplus k)$ and

$$\widehat{\varphi}_{0,k}(\omega) = w_k(\omega)\widehat{\varphi}(\omega) = w_k(\omega)\mathbf{1}_{[0,t)}(\omega), \quad k \in \mathbb{Z}_+.$$

Therefore, an application of the inverse Walsh-Fourier transform shows that $\{\varphi(\cdot \oplus k) \mid k \in \mathbb{Z}_+\}$ is a Parseval frame for $V_0(t)$. Also, in view of (4.9),

$$V_j(t) = D^j(V_0(t)), \quad \text{where } Df(x) = 2^{1/2} f(2x).$$

Observing that $\varphi_{j,k} = D^j \varphi_{0,k}$, we conclude that for each $j \in \mathbb{Z}$ the system $\{\varphi_{jk} \mid k \in \mathbb{Z}_+\}$ is a Parseval frame for $V_j(t)$. From this, since $\varphi \in V_0(t) \subset V_1(t)$, by Proposition 4.1 it follows that

$$\varphi(x) = \sum_{k \in \mathbb{Z}_+} \langle \varphi, \varphi_{1k} \rangle \varphi_{1k}(x), \quad x \in \mathbb{R}_+. \tag{4.11}$$

Also, by (4.1) and Parseval's formula we have

$$\begin{aligned} \langle \varphi, \varphi_{1k} \rangle &= \sqrt{2} \int_{\mathbb{R}_+} \varphi(x) \varphi(2x \oplus k) dx \\ &= \frac{1}{\sqrt{2}} \int_0^t \widehat{\varphi}(\omega) \widehat{\varphi}(\omega/2) w_k(\omega/2) d\omega \\ &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}_+} \widehat{\varphi}(\omega) \chi(k/2, \omega) d\omega \\ &= \frac{1}{\sqrt{2}} \varphi(k/2), \end{aligned}$$

which by (4.11) yields (4.10). □

Remark 4.7. For $0 < t < 1$ let us denote by $W_j(t)$ the orthogonal complement of $V_j(t)$ in $V_{j+1}(t)$:

$$W_j(t) = V_j(t)^{\perp} \cap V_{j+1}(t), \quad j \in \mathbb{Z}.$$

Then the orthogonal projection $Q_j : L^2(\mathbb{R}_+) \to W_j(t)$ can be defined as follows:

$$g_j = Q_j f \iff \widehat{g}_j = \widehat{f} \cdot \mathbf{1}_{[2^j t, 2^{j+1} t)}, \quad f \in L^2(\mathbb{R}_+).$$

Hence, letting $\psi = \mathscr{D}_{2t} - \mathscr{D}_t$, we see that

$$\widehat{\psi}_{jk}(\omega) = 2^{-j/2} w_k(\omega) \mathbf{1}_{[2^j t, 2^{j+1} t)}(\omega), \quad j \in \mathbb{Z}, \;\; k \in \mathbb{Z}_+,$$

and that, for each j, the system $\{\psi_{jk} \mid k \in \mathbb{Z}_+\}$ is a Parseval frame for $W_j(t)$.

Remark 4.8. Suppose that $f \in V_j(t)$, where $j \in \mathbb{Z}$, $0 < t \leq 1$. Using Theorem 4.5 and Proposition 4.1, we then obtain

$$f(x) = \sum_{k \in \mathbb{Z}_+} \langle f, \varphi_{jk} \rangle \varphi_{jk}(x), \quad x \in \mathbb{R}_+,$$

where $\langle f, \varphi_{jk} \rangle = 2^{-j/2} f(k2^{-j})$, $\varphi_{jk}(x) = 2^{j/2} \mathscr{D}_t(2^j x \oplus k)$. Thus, the following formula holds for every $f \in V_j(t)$:

$$f(x) = \sum_{k=0}^{\infty} f(k2^{-j}) \mathscr{D}_t(2^j x \oplus k), \quad x \in \mathbb{R}_+.$$

Notice that, in view of (4.7), for $t = 1$ this gives (4.6).

We say that a compactly supported function $\varphi \in L^2(\mathbb{R}_+)$ is a *refinable function*, if it satisfies an equation of the type

$$\varphi(x) = \sum_{k=0}^{2^n-1} c_k \varphi(2x \oplus k), \quad x \in \mathbb{R}_+. \tag{4.12}$$

Using the Walsh-Fourier transform, we obtain $\widehat{\varphi}(\omega) = m_0(\omega/2)\widehat{\varphi}(\omega/2)$, where the Walsh polynomial

$$m_0(\omega) = \frac{1}{2} \sum_{k=0}^{2^n-1} c_k w_k(\omega)$$

is the *mask* of equation (4.12). Note that $m_0 \in \mathscr{E}_n$ and $m_0(\omega+1) = m_0(\omega)$ for all $\omega \in \mathbb{R}_+$. Moreover, the coefficients of equation (4.12) are related to the values

$$b_l = m_0(\omega), \quad \omega \in [l2^{-n}, (l+1)2^{-n}), \quad 0 \le l \le 2^n - 1,$$

by means of the discrete Walsh transform:

$$c_k = \frac{1}{2^{n-1}} \sum_{l=0}^{2^n-1} b_l w_l(k2^{-n}), \quad 0 \le k \le 2^n - 1;$$

cf. (2.2) and (2.3). Suppose that M either is the union of some of the intervals $[l2^{-n}, (l+1)2^{-n}), \quad l = 1, 2, \dots, 2^n - 1$, or coincides with one of these intervals. Then we define

$$S(M) := \{\omega/2 : \ \omega \in M\} \cup \{(\omega+1)/2 : \omega \in M\}.$$

A set M is said to be *blocking* for a mask m_0 if it satisfies the condition

$$S(M) \subset M \cup \{\omega \in \Delta |\ m_0(\omega) = 0\}.$$

According to [3] we have the following theorem:

Theorem 4.9. *Let $\varphi \in L^2(\mathbb{R}_+)$ be a compactly supported solution of equation (4.12) such that $\widehat{\varphi}(0) = 1$. Suppose that the mask m_0 satisfies*

$$|m_0(\omega)|^2 + |m_0(\omega + 1/2)|^2 = 1 \quad \textit{for all} \ \ \omega \in [0, 1/2). \tag{4.13}$$

Then the following are equivalent:

(i) *The function φ generates a multiresolution analysis in $L^2(\mathbb{R}_+)$.*
(ii) *The mask m_0 has no bloking sets.*

Under the hypotheses of Theorem 4.8 the orthogonal wavelet ψ in $L^2(\mathbb{R}_+)$ can be defined (see [3]) by the formula:

$$\psi(x) = \sum_{k=0}^{2^n-1} (-1)^k \bar{c}_{k \oplus 1} \varphi(2x \oplus k), \quad x \in \mathbb{R}_+. \tag{4.14}$$

Combining this result with Theorem 4.4 and Theorem 4.5, we can construct frames for $L^2(\mathbb{R}_+)$. Let us illustrate this by the following example:

EXAMPLE 4. The scaling function φ, which was introduced in [13], is a solution of equation (4.12) with $n = 2$ and

$$c_0 = \frac{1+a+b}{2}, \quad c_1 = \frac{1+a-b}{2}, \quad c_2 = \frac{1-a-b}{2}, \quad c_3 = \frac{1-a+b}{2}, \qquad (4.15)$$

where $0 < |a| < 1$, $|b| = \sqrt{1-|a|^2}$. This function generates a multiresolution analysis in $L^2(\mathbb{R}_+)$, possesses the self-similarity property:

$$\varphi(x) = \begin{cases} (1+a-b)/2 + b\varphi(2x), & 0 \le x < 1, \\ (1-a+b)/2 - b\varphi(2x-2), & 1 \le x \le 2 \end{cases}$$

and it is represented by a lacunary Walsh series:

$$\varphi(x) = (1/2)\mathbf{1}_{[0,1)}(x/2)(1 + a\sum_{j=0}^{\infty} b^j w_{2^{j+1}-1}(x/2)), \quad x \in \mathbb{R}_+.$$

In the case where $a = 1$, $b = 0$ the Haar function $\phi = \mathbf{1}_{[0,1)}$ is obtained from (4.12) and (4.15). On the other hand, for $\varphi = \mathscr{D}_{1/2}$, by Theorem 4.5 we have

$$\varphi(x) = \frac{1}{2}\sum_{k=0}^{3} \varphi(2x \oplus k), \quad x \in \mathbb{R}_+.$$

Observing that $\varphi(x) = 2^{-1}\mathbf{1}_{[0,1)}(x/2)$ and $\varphi(x) = \varphi(x \oplus 1)$, we may write $\varphi(x) = \varphi(2x) + \varphi(2x \oplus 3)$, which corresponds to the values $a = 0$ and $b = 1$ in (4.15). Then the interval $[1/2, 1)$ is a blocking set. Furthermore, if $\varphi = \mathscr{D}_{1/2}$, we obtain formally from (4.14) that $\psi(x) = -\varphi(2x \oplus 1) + \varphi(2x \oplus 2)$. Since

$$\widehat{\psi}(\omega) = 2^{-1}\mathbf{1}_{[0,1)}(\omega)[w_2(\omega/2) - w_1(\omega/2)] = -\mathbf{1}_{[1/4,1/2)}(\omega),$$

we get by Theorem 4.4 that ψ generates a Parseval frame for $L^2(\mathbb{R}_+)$.

Remark 4.10. Suppose that $\varphi \in L^2(\mathbb{R}_+)$ is a compactly supported solution of equation (4.12), φ is continuous at 0 and $\widehat{\varphi}(0) \neq 0$. As above, assume that m_0 satisfies (4.13) and ψ is defined by (4.14). Then $\{\psi_{jk}\}$ is a tight frame for $L^2(\mathbb{R}_+)$ with frame bound $|\widehat{\varphi}(0)|^2$ (cf. [1], [12]). Moreover, if m_0 has a blocking set, then $\{\psi_{jk}\}$ is not a basis in $L^2(\mathbb{R}_+)$.

References

[1] M. Bownik, Tight frames of multidimensional wavelets, *J. Fourier Anal. Appl.* **3**(1997), 525–542.

[2] C.K. Chui and H.N. Mhaskar, On trigonometric wavelets, *Constr. Approx.* **9**(1993), 167–190.

[3] Yu.A. Farkov, On wavelets related to the Walsh series, *J. Approx. Theory* **161**(1)(2009), 259–279.

[4] Yu.A. Farkov, Biorthogonal wavelets on Vilenkin groups, *Proceedings of the Steklov Institute of Mathematics* **265**(2009), 101–114.

[5] Yu.A. Farkov and E.A. Rodionov, Estimates of the smoothness of dyadic orthogonal wavelets of Daubechies type, *Math. Notes* **86:3**(2009), 392–406.

[6] Yu.A. Farkov, Walsh functions and continuous wavelet transform, in : *Proceedings of 5th International Symposium "Fourier Series and Their Applications"* (May 27–June 3, 2008), Izd. TsVVR, Rostov-on-Don, 27–32 (2008) (in Russian).

[7] F.A. Shah, Construction of wavelet packets on p-adic field, *Intern. J. Wavelets, Multiresolution and Information Processing* **5**(2009), 553–565.

[8] B. Fisher and J. Prestin, Wavelets based on orthogonal polynomials, *Math. Computation* **66**(220)(1997), 1593–1618.

[9] B.I. Golubov, *Elements of Dyadic Analysis*, LKI, Moscow (2007) (in Russian).

[10] B.I. Golubov, A.V. Efimov and V.A. Skvortsov, *Walsh Series and Transforms: Theory and Applications*, LKI, Moscow (2008); Klumer, Dordrecht (1991).

[11] D. Han, K. Kornelson, D. Larson and E. Weber, *Frames for Undergraduates*, AMS (*Student Mathematical Library*) **40** (2007).

[12] W. Lawton, Tight frames of compactly supported affine wavelets, *J. Math. Phis.* **31**(1990), 1898–1901.

[13] W.C. Lang, Orthogonal wavelets on the Cantor dyadic group, *SIAM J. Math. Anal.* **27**(1)(1996), 305–312.

[14] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, New York-London (1999).

[15] I.Ya. Novikov, V.Yu. Protassov and M.A. Skopina, *Wavelet Theory*, Fizmatlit, Moscow (2006) (in Russian).

[16] F. Schipp, W.R. Wade and P. Simon, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, New York (1990).

[17] L.A. Zalmanzon, *Fourier, Walsh and Haar Transforms and Their Application in Control, Communication and Other Domains*, Nauka, Moscow (1989) (in Russian).

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