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Research Article

Best Proximity Point Results for Quasi Contractions of Perov Type in Non-Normal Cone Metric Space

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Abstract. In this paper, we study the notion of Ćirić-Perov quasi contraction and Fisher-Perov quasi contraction and prove some best proximity point theorems for such contractions in the frame work of non-normal regular cone metric spaces. We give an example to support our result. Our results extend and generalized many existing results in literature.

Keywords. Cone metric spaces; Non-normal cones; Best proximity point; Perov contraction; Spectral radius

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1. Introduction

In 2007, cone metric spaces were introduced by Huang and Zhang, as a generalization of metric spaces where they investigate the convergence and completeness in cone metric spaces and proved some fixed point theorems for contractive mappings on these spaces [16]. They gave the analogue of Banach contraction principle and other basic theorems in the framework of cone metric spaces. Later on, by omitting the assumption of normality in the results of [16], Rezapour and Hamlbarani [29] obtained some fixed point theorems, as the generalizations of the results presented in [16]. Since then, many authors have been interested in the study of fixed point results in the setting of cone metric spaces (see [3], [5], [7], [14], [17], [18], [20], [28], [29]).

In 1974, Ćirić [8] introduced the notion of quasi-contraction as one of the most general contractive type maps and prove the related fixed point theorem. Ilić [18] define and study quasi-contraction on a cone (normal) metric space and proved a fixed point theorem which generalized the results of Guang and Xian [16] and Ćirić [8]. Kadelburg *et al.* [20] obtained a fixed point result for quasi contraction with contractive constant $\lambda \in \left(0, \frac{1}{2}\right)$ without using the normality condition. Recently, Gajić and Rakočević [14] obtained a similar result for contractive constant $\lambda \in (0,1)$. Cvetković and Rakočević [9, 10] introduced the notions of quasi contraction of Perov type and Fisher quasi contraction of Perov type and proved fixed point theorems under these contractions.

Kirk *et al.* [21] generalized the Banach contraction principle by using two closed subsets of a complete metric space. Later on, Petrusel [27] proved some results about periodic points of cyclic contraction maps as a generalization of Kirk's main result. In 2006, Eldered and Veeramani [13] proved some results about best proximity points of cyclic contraction maps. Basha [6] obtained best proximity point theorems for non-self proximal contractions in complete metric space. Abbas *et al.* [1] proved the existence and uniqueness of coincidence best proximity point under proximal cyclic contractions of Perov type in the frame work of regular cone metric space. For more results concerning best proximity point theory, we refer to [2, 4, 22, 25, 26] and the references therein.

Recently, Hagi *et al.* [15] defined the notion of distance between two subsets in regular cone metric spaces and established some conditions which gives the existence of best proximity points for cyclic contraction mappings on regular cone metric spaces.

The aim of this paper is to prove some best proximity point theorems for Ćirić-Perov quasi contraction and Fisher-Perov quasi contraction with non-normal cone. Example is given to support our result. Our results generalized the main results of Cvetković and Rakočević [9, 10].

2. Preliminaries

Let E be a real Banach space. A subset P of E is called a cone if

(i) P is nonempty, closed and $P \neq \{\theta\}$ (where θ is the zero element of E);

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ implies that $ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

Partial ordering on E is defined with the help of a cone P as follows:
For $x, y \in E$, $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ and $x \ll y$ stands for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. If $\operatorname{int} P$ is non-empty then P is called a solid cone. A cone P is normal if there is a number $K > 0$ such that for all

$x,y \in P$,

$$\theta \preceq x \preceq y \quad \text{implies that} \quad \|x\| \le K\|y\|. \tag{1}$$

The least positive number satisfying the above inequality is called a normal constant of P. A cone P is called regular if every bounded above increasing sequence in E is convergent, or equivalently a cone P is regular if every decreasing sequence which is bounded below is convergent. It is known that every regular cone is normal [29].

Definition 2.1. Let X be a nonempty set. A mapping $d : X \times X \to E$ is said to be a cone metric on X if for any $x,y,z \in X$, the following conditions hold:

(d1) $\theta \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$;

(d3) $d(x,y) \preceq d(x,z) + d(y,z)$.

The pair (X,d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space. Furthermore, the category of regular cone metric spaces is bigger then the category of metric spaces ([15, Example 1.1]).

A sequence $\{x_n\}$ in a cone metric space (X,d) is called: Cauchy sequence if there is an N such that $d(x_n,x_m) \ll c$ for all $n,m > N$. Convergent if there exist an N and $x \in X$ such that $d(x_n,x) \ll c$ for all $n > N$. The limit of a convergent sequence is unique. A sequence $\{x_n\}$ is Cauchy if $d(x_n,x_m) \to 0$ as $n,m \to \infty$. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. If the cone is normal then a sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $d(x_n,x) \to 0$ as $n \to \infty$. For further details of these properties, we refer to ([9], [14], [16], [17], [19], [28]). A subset A of X is closed if and only if every convergent sequence in A has its limit in A.

Throughout this paper, (X,d) is a regular cone metric space, A and B nonempty subsets of X.

(i) If $c \gg 0$, then there exists $\delta > 0$ such that $\| b \| < \delta$ implies $b \ll c$.

(ii) For any given $c \gg 0$ and $c_0 \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $\frac{c_0}{n_0} \ll c$.

(iii) If $\{a_n\},\{b_n\}$ are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \le b_n$ for all $n \ge 1$, then $a \le b$.

We write $\mathscr{B}(E)$ for the set of all bounded linear operators on E and $L(E)$ for the set of all linear operators on E. $\mathscr{B}(E)$ is a Banach algebra, and if $\mathscr{A} \in \mathscr{B}(E)$, let

$$r(\mathscr{A}) = \lim_{n\to\infty} \left\|\mathscr{A}^n\right\|^{\frac{1}{n}} = \inf_n \left\|\mathscr{A}^n\right\|^{\frac{1}{n}} \tag{2}$$

be the spectral radius of $\mathscr{A}$. We write $\mathscr{B}(E)^{-1}$ for the set of all invertible elements in $\mathscr{B}(E)$. Let us remark that if $r(\mathscr{A}) < 1$, then

1. Series $\sum_{n=0}^{\infty} \mathscr{A}^n$ is absolutely convergent;
2. $I - \mathscr{A}$ is invertible in $\mathscr{B}(E)$ and

$$\sum_{n=0}^{\infty} \mathscr{A}^n = (I - \mathscr{A})^{-1}. \tag{3}$$

Let E be a real Banach space, $P \subseteq E$ cone in E and $\mathscr{A} : E \to E$ a linear operator. The following conditions are equivalent: $\mathscr{A}$ is increasing, that is, $x \preceq y$ implies that $\mathscr{A}(x) \preceq \mathscr{A}(y)$; if and only if $\mathscr{A}$ is positive, that is, $\mathscr{A}(P) \subset P$.

Cvetković [9, 10] obtained the following generalizations of Banach Contraction Principle:

Definition 2.2 ([9])**.** Let (X,d) be a cone metric space. A map $f : X \to X$ such that for some bounded linear operator $\mathscr{A} \in \mathscr{B}(E)$, $\rho(\mathscr{A}) < 1$ and for each $x, y \in X$, there exists

$$u \in C(f,x,y) \equiv \{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\},$$

such that

$$d(fx, fy) \preceq \mathscr{A}(u),$$

is called a quasi-contraction of Perov type.

Theorem 2.3 ([9])**.** *Let (X,d) be a complete cone metric space with a solid cone P. If a mapping $f : X \to X$ is a quasi-contraction and $\mathscr{A}(P) \subseteq P$, then f has a unique fixed point $x^* \in X$ and, for any $x \in X$, the iterative sequence $(f^n x)_{n\in\mathbb{N}}$ converges to the fixed point of f.*

Definition 2.4 ([10])**.** Let (X,d) be a cone metric space. A map $f : X \to X$ such that for some $\mathscr{A} \in B(E)$, $r(\mathscr{A}) < 1$ and for some fixed positive integers p and q, and for every $x, y \in X$, there exists

$$u \in F_f^{p,q}(x,y) \equiv \{d(f^r x, f^s y), d(f^r x, f^{r'} x), d(f^s y, f^{s'} y)\},$$

where $0 \le r$, $r' \le p$ and $0 \le s$, $s' \le q$, such that

$$d(f^p x, f^q y) \preceq \mathscr{A}(u). \tag{4}$$

is called (p,q)-quasi-contraction (Fisher's quasi-contraction, F quasi-contraction) of Perov type.

Theorem 2.5 ([10])**.** *Let (X,d) be a complete cone metric space and P be a cone with int $P \neq \phi$. Suppose the mapping $f : X \to X$ is a (p,q)-quasi-contraction of Perov type, $\mathscr{A}(P) \subseteq P$ and let f be continuous. Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Set $\Delta = \{p \in P : p \preceq d(x,y)$ for all $x \in A, y \in B\}$. Obviously this set is nonempty as $\theta \in \Delta$. We denote a unique vector $p \in \triangle$ by $dis(A,B) \equiv d(A,B)$ if for any q in $\triangle$, we have $q \preceq p$ [1]. We also define

$$A_0 = \{x \in A : d(x,y) = p \text{ for some } y \in B\} \text{ and } B_0 = \{y \in B : d(x,y) = p \text{ for some } x \in A\}.$$

3. Ćirić-Perov Quasi Contraction

In this section we present the notion of Ćirić-Perov quasi contraction and prove best proximity point result for such contraction with an example.

Lemma 3.1 ([9])**.** *Let (X,d) be a cone metric space. Suppose that $\{x_n\}$ is a sequence in X and that b_n is a sequence in E. If $0 \le d(x_n, x_m) \le b_n$ for $m > n$ and $b_n \to 0$ as $n \to \infty$, then x_n is a Cauchy sequence.*

Lemma 3.2 ([9])**.** *Let E be Banach space, $P \subseteq E$ cone in E and $\mathscr{A} : E \to E$ linear operator. The following conditions are equivalent:*

(i) *$\mathscr{A}$ is increasing, i.e., $x \le y$ implies $\mathscr{A}(x) \le \mathscr{A}(y)$;*

(ii) *$\mathscr{A}$ is positive, i.e., $\mathscr{A}(P) \subset P$.*

Definition 3.1. Let A and B be closed subsets of a cone metric space (X,d), and let $f:A\to B$. Then, f is called Ćirić-Perov quasi contraction if for some operator $\mathscr{A}\in\mathscr{B}(E)$, $r(\mathscr{A})<1$ and for every $x,y\in A$, there exists

$$u\in C(f;x,y)\equiv\{d(x,y)-p,d(x,fx)-p,d(y,fy)-p,d(x,fy)-p,d(y,fx)-p\},$$

such that

$$d(fx,fy)\le\mathscr{A}(u). \tag{5}$$

Theorem 3.2. *Let (X,d) be a complete cone metric space with a solid cone P, $\operatorname{int}(P)\neq\phi$ and for $\mathscr{A}\in\mathscr{B}$ (E), $\mathscr{A}(P)\subseteq P$. Let $f:A\to B$ be continuous Ćirić-Perov quasi contraction with $f(A)\subseteq B$. Then f has a unique best proximity point x^* in X.*

Proof. First we will prove the following two inequalities for any $x\in X$:

(i) $d(f^n(x),f(x))\le(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)$, $n\in\mathbb{N}$,

(ii) $d(f^n(x),x)\le(\mathscr{I}-\mathscr{A})^{-1}(d(f(x),x)-p)$, $n\in\mathbb{N}$.

Inequality (i) is true for $n=1$. Suppose that it's satisfied for each $m\le n$.
Since $d(f^{n+1}(x),f(x))\le\mathscr{A}(u)$, where

$$u\in\{d(f^n(x),x)-p,d(f^n(x),f(x))-p,d(x,f(x))-p,d(x,f^{n+1}(x))-p,d(f^n(x),f^{n+1}(x))-p\},$$

we will consider five cases.

Case I: If $u=d(f^n(x),x)-p$, then

$$\begin{aligned}d(f^{n+1}(x),f(x))&\le\mathscr{A}(d(f^n(x),x)-p)\le\mathscr{A}(d(f^n(x),f(x)))+\mathscr{A}(d(f(x),x)-p)\\&\le\mathscr{A}(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)+\mathscr{A}(d(f(x),x)-p)\\&=(-(\mathscr{I}-\mathscr{A})+\mathscr{I})(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)+\mathscr{A}(d(f(x),x)-p)\\&=-\mathscr{A}(d(f(x),x)-p)+(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)+\mathscr{A}(d(f(x),x)-p)\\&=(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)\end{aligned}$$

$$\Rightarrow\qquad(f^{n+1}(x),f(x))\le(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p).$$

Case II: If $u=d(f^n(x),f(x))-p$, then

$$\begin{aligned}d(f^{n+1}(x),f(x))&\le\mathscr{A}(d(f^n(x),f(x))-p)\le\mathscr{A}((\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)-p)\\&=(\mathscr{I}-(\mathscr{I}-\mathscr{A}))((\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)-p)\\&=(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)-\mathscr{A}(d(f(x),x)-p)-\mathscr{A}(p)\\&\le(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)\end{aligned}$$

$$\Rightarrow\qquad(f^{n+1}(x),f(x))\le(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p).$$

Case III: If $u=d(f(x),x)-p$, then

$$\begin{aligned}d(f^{n+1}(x),f(x))&\le\mathscr{A}(d(f(x),x)-p)\\&\le(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)\end{aligned}$$

$$\Rightarrow\qquad(f^{n+1}(x),f(x))\le(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p).$$

Case IV: If $u=d(x,f^{n+1}(x))-p$. Using the triangle inequality,

$$d(x,f^{n+1}(x))-p\le(d(x,f(x))+d(f(x),f^{n+1}(x)))-p,$$

and the fact that $\mathscr{A}(P)\subseteq P$, we have

$$d(f^{n+1}(x),f(x))\le\mathscr{A}(d(x,f(x))-p)+\mathscr{A}(d(f(x),f^{n+1}(x))),$$

hence,

$$d(f^{n+1}(x),f(x)) \le (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(x,f(x))-p).$$

Case V: If $u = d(f^n(x), f^{n+1}(x)) - p$, then

$$d(f^{n+1}(x),f(x)) \le \mathscr{A}(d(f^n(x),f^{n+1}(x))-p)$$

and since f is a Ćirić-Perov quasi-contraction, we have

$$d(f^n(x),f^{n+1}(x))-p \le \mathscr{A}^{n-1+i}(d(f(x),f^j(x))-p),$$

for some $i \in \{0,1,2,\dots,n\}$, $j \in \{1,2,n+1\}$. The case where $j = n+1$, implies $d(f^{n+1}(x),f(x)) = 0$. Indeed, since $\mathscr{I}-\mathscr{A}^{n+i}$ is an invertible operator and $\mathscr{A}^{n+i}(P) \subseteq P$, we have

$$d(f^{n+1}(x),f(x)) \le (\mathscr{I}-\mathscr{A}^{n+i})^{-1}(0) = 0,$$

therefore $d(f^{n+1}(x),f(x)) = 0$.

On the other hand

$$\begin{aligned} d(f^{n+1}(x),f(x)) &\le \mathscr{A}^{n+i}(d(f(x),f^j(x))-p) \\ &\le \mathscr{A}^{n+i}(\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p) \\ &\le (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p), \end{aligned}$$

implies

$$d(f^{n+1}(x),f(x)) \le (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p).$$

Hence, the inequality (i) holds for $n \in \mathbb{N}$.

Now we prove the inequality (ii) by using inequality (i) as follows:

$$\begin{aligned} d(f^n(x),x) &\le d(f^n(x),f(x)) + d(f(x),x) \\ &\le (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p) + d(f(x),x) \\ &= (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)) - (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(p) + d(f(x),x) \\ &= (\mathscr{I}-\mathscr{A})^{-1}(d(f(x),x)) - (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(p) \\ &= (\mathscr{I}-\mathscr{A})^{-1}(d(f(x),x) - \mathscr{A}(p)) \end{aligned}$$

$n \in \mathbb{N}$. We prove that $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in A. Suppose that $n,m \in \mathbb{N}$, $m > n$. Since f is Ćirić-Perov quasi contraction, so there exist $i,j \in \mathbb{N}$, $1 \le i \le n$, $1 \le j \le m$ such that

$$\begin{aligned} d(f^n(x),f^m(x)) &\le \mathscr{A}^{n-1}(d(f^i(x),f^j(x))-p) \\ &\le \mathscr{A}^{n-1}(d(f^i(x),f(x)) + d(f(x),f^j(x))-p) \\ &\le \mathscr{A}^{n-1}((\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p) + (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(d(f(x),x)-p)-p) \\ &\le 2\mathscr{A}^n(\mathscr{I}-\mathscr{A})^{-1}(d(f(x),x)-p) \end{aligned}$$

this implies

$$d(f^n(x),f^m(x)) \le 2\mathscr{A}^n(\mathscr{I}-\mathscr{A})^{-1}(d(f(x),x)-p).$$

Since $2\mathscr{A}^n(\mathscr{I}-\mathscr{A})^{-1}(d(f(x),x)-p) \to 0$, $n \to \infty$, by Lemma 3.1, $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists $x^* \in A$ such that $\lim_{n\to\infty} f^n(x) = x^*$.

Suppose that $c \ll 0$ and $\epsilon \gg 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$d(x^*,f^n(x)) \ll c, d(f^n(x),f^m(x)) \ll \epsilon \text{ and } d(x^*,f^n(x)) \ll \epsilon \text{ for all } n,m \ge n_0. \tag{6}$$

Now, for each $n > n_0$,

$$d(x^*,f(x^*)) - p \ll d(x^*,f^n(x)) + d(f^n(x),f(x^*)) - p \le c + d(f(x^*),f^n(x)) - p. \tag{7}$$

Since f is a Ćirić-Perov quasi contraction, we have

$$d(f^n(x), f(x^*)) \leq \mathscr{A}(u) \tag{8}$$

where

$u \in \{d(f^{n-1}(x), x^*) - p, d(f^{n-1}(x), f^n(x)) - p, d(f^{n-1}(x), f(x^*)) - p, d(x^*, f(x^*)) - p, d(f(x^*), f^n(x)) - p\}$.

If

$$u \in \{d(f^{n-1}(x), x^*) - p, d(f^{n-1}(x), f^n(x)) - p, d(f(x^*), f^n(x)) - p\},$$

for infinitely many $n > n_0$, then, by (6), (7) and (8), we have

$$d(x^*, f(x^*)) - p \leq c + \mathscr{A}(\epsilon). \tag{9}$$

Since the inequality (9) is true for each $c \gg 0$, we get

$$d(x^*, f(x^*)) - p \leq \mathscr{A}(\epsilon). \tag{10}$$

If $u = d(f^{n-1}(x), f(x^*)) - p$, then

$$d(f^{n-1}(x), f(x^*)) - p \leq d(f^{n-1}(x), x^*) + d(x^*, f(x^*)) - p$$

and $\mathscr{A}(P) \subseteq P$ implies

$$\mathscr{A}(u) \leq \mathscr{A}(d(f^{n-1}(x), x^*)) + \mathscr{A}(d(x^*, f(x^*)) - p).$$

Again by inequalities (6), (7) and (8), we have

$$d(x^*, f(x^*)) - p \leq c + \mathscr{A}(\epsilon) + \mathscr{A}(d(x^*, f(x^*) - p),$$

since $c \gg 0$ is arbitrary, it follows

$$d(x^*, f(x^*) - p \leq \mathscr{A}(\epsilon) + \mathscr{A}(d(x^*, f(x^*) - p)$$

implies

$$(\mathscr{I} - \mathscr{A})(d(x^*, f(x^*)) - p) \leq \mathscr{A}(\epsilon). \tag{11}$$

However, $(\mathscr{I} - \mathscr{A})^{-1}$ is increasing, so inequality (11) implies

$$d(x^*, f(x^*) - p \leq (\mathscr{I} - \mathscr{A})^{-1}\mathscr{A}(\epsilon). \tag{12}$$

Finally, if $u = d(x^*, f(x^*)) - p$, then by (7) and (8)

$$(\mathscr{I} - \mathscr{A})(d(x^*, f(x^*) - p) \leq c. \tag{13}$$

From (13), we conclude that

$$d(x^*, f(x^*)) - p \leq (\mathscr{I} - \mathscr{A})^{-1}(c). \tag{14}$$

Now, by (10), (12) and (14), for $\epsilon = \frac{\epsilon}{n}$ and $c = \frac{c}{n}$, $n = 1, 2, \dots$, we get, respectively,

$$0 \leq d(x^*, f(x^*)) - p \leq \mathscr{A}\left(\frac{\epsilon}{n}\right) = \frac{\mathscr{A}(\epsilon)}{n} \to 0, n \to \infty,$$

$$0 \leq d(x^*, f(x^*)) - p \leq (\mathscr{I} - \mathscr{A})^{-1}\mathscr{A}\left(\frac{\epsilon}{n}\right) = \frac{(\mathscr{I} - \mathscr{A})^{-1}\mathscr{A}(\epsilon)}{n} \to 0, \quad n \to \infty$$

and

$$0 \leq d(x^*, f(x^*)) - p \leq (\mathscr{I} - \mathscr{A}_1)^{-1}\left(\frac{c}{n}\right) = \frac{(\mathscr{I} - \mathscr{A})^{-1}(c)}{n} \to 0, \quad n \to \infty.$$

Therefore, $d(x^*, f(x^*)) - p = 0$ gives $d(x^*, f(x^*)) = p$.

For uniqueness, let y^* be another best proximity point of f. i.e. $d(y^*, fy^*) = p$, then

$$\begin{aligned} d(x^*, y^*) &\leq d(x^*, f(x^*)) + d(f(x^*), f(y^*)) + d(f(y^*), y^*) \\ &\leq 2p + \mathscr{A}(d(x^*, y^*) - p) \\ &= 2p + \mathscr{A}(d(x^*, y^*)) - \mathscr{A}(p) \end{aligned}$$

$$\leq 2p + \mathscr{A}(d(x^*, y^*)) - p$$
$$= p + \mathscr{A}(d(x^*, y^*))$$

implies

$$(\mathscr{I} - \mathscr{A})(d(x^*, y^*)) \leq p,$$

which necessitates that $p > 0$. This completes the proof. □

Example 3.1. Let $X = \mathbb{R}$ and $E = C^1[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on a non-normal cone $P = \{x \in E : x(t) \geq 0 \text{ on } [0,1]\}$. Let $A = [0,7]$ and $B = [9,12]$, then $d(A,B) = exp(2)$. Define a cone metric $d : X \times X \to E$ by

$$d(x, y) = \exp(|x - y|), \quad x, y \in X,$$

$f : A \to B$ by

$$f(x) = \begin{cases} 12 & x \in [0,3] \\ 9 & x \in (3,7]. \end{cases}$$

Clearly, $f(A) \subseteq B$. Considering the case for $x \in [0,3]$ and $y = (3,7]$, then $d(fx, fy) = \exp(3)$, $d(y, fx) \geq \exp(5)$ and $d(x, fy) \geq \exp(6)$. Hence

$$d(fx, fy) = \exp\left(\frac{3}{5} \cdot 5\right)$$
$$\leq \exp\left(\frac{3}{5} \cdot \max\{|x - fy| - d(A,B), |y - fx| - d(A,B)\}\right).$$

Thus f is a Ćirić-Perov quasi contraction with a bounded linear operator $\mathscr{A} : E \to E$ defined by $\mathscr{A}(f) = \left(\frac{3}{5}\right)$. Clearly $\|\mathscr{A}\| = \frac{3}{5}$. Hence all the conditions of Theorem 3.2 are satisfied. Thus, f has a unique best proximity point $x = 7 \in A$.

4. Fisher-Perov Quasi Contraction

In this section, we define the notion of Fisher-Perov quasi contraction and prove the existence and uniqueness of best proximity point under such contraction.

Definition 4.1. Let A and B be closed subsets of a cone metric space (X,d), and let $f : A \to B$. Then, f is called Fisher-Perov quasi ((l,m)-quasi) contraction, if for some $\mathscr{A} \in \mathscr{B}(E)$, $r(\mathscr{A}) < 1$, for some fixed positive integers l and m and for every $x, y \in A$, there exists an element

$$u \in F_f^{l,m}(x, y) \equiv \{d(f^r x, f^s y) - p, d(f^r x, f^{r'} x) - p, d(f^s y, f^{s'} y) - p\},$$

where $0 \leq r$, $r' \leq l$ and $0 \leq s$, $s' \leq m$, such that

$$d(f^l x, f^m y) \leq \mathscr{A}(u). \tag{15}$$

Theorem 4.2. *Let (X,d) be a complete cone metric space with a solid cone P, $\operatorname{int}(P) \neq \phi$, A and B closed subsets of X. Let $f : A \to B$ be continuous Fisher-quasi contraction, $\mathscr{A}(P) \subseteq P$ and $f(A) \subseteq B$. Then f has a unique best proximity point x^* in A.*

Proof. Assume that $l \geq m$. For an arbitrary $x \in A$, we first prove

$$d(f^n(x), f^l(x)) \leq \mathscr{A}(1 - \mathscr{A})^{-1}(u(x) - p), \quad n \geq l, \tag{16}$$

where

$$u(x) = \sum_{0 \leq i < l} d(f^i(x), f^l(x)). \tag{17}$$

Inequality (16) is true for $n = 1,2,\dots,l$. Suppose that it is true for each $m \le n_0 - 1$, we prove (16) for $m = n_0 \ge l+1$.
Since f is Fisher-Perov quasi contraction, there exists $i,j \in \mathbb{N}$, such that

$$d(f^{n_0}(x), f^l(x)) \le \mathscr{A}(d(f^i x, f^j x) - p). \tag{18}$$

Case I: If $i,j \le l$, then

$$\begin{aligned} d(f^{n_0}(x), f^l(x)) &\le \mathscr{A}(d(f^i x, f^j x) - p) \\ &\le \mathscr{A}[d(f^i x, f^l x) + d(f^l x, f^j x) - p] \\ &\le \mathscr{A}(u(x) - p) \\ &\le (\mathscr{I} - \mathscr{A})^{-1}\mathscr{A}(u(x) - p). \end{aligned}$$

Case II: If $l < i < n_0$, $j \le l$, then from (16) and (18), we have

$$\begin{aligned} d(f^{n_0}(x), f^l(x)) &\le \mathscr{A}(d(f^i x, f^j x) - p) \\ &\le \mathscr{A}[d(f^l x, f^j x) + (d(f^i x, f^l x) - p)] \\ &\le \mathscr{A}(\mathscr{I} - \mathscr{A})^{-1}\mathscr{A}(u(x) - p) + \mathscr{A}(u(x) - p) \\ &= (1 - (1 - \mathscr{A}))((\mathscr{I} - \mathscr{A})^{-1}\mathscr{A})(u(x) - p) + \mathscr{A}(u(x) - p) \\ &= \mathscr{A}(\mathscr{I} - \mathscr{A})^{-1}(u(x) - p). \end{aligned}$$

Case III: If $l < i < n_0$, $l < j < n_0$, we have

$$d(f^{n_0}(x), f^l(x)) \le \mathscr{A}^k(d(f^{i_0} x, f^{j_0} x) - p), \tag{19}$$

where $i_0 < l$ or $j_0 < l$ and $1 < k$. Assume that at least $i_0 < l$.

$$\begin{aligned} d(f^{n_0}, f^l x) &\le \mathscr{A}^k(d(f^{i_0} x, f^{j_0} x) - p) \\ &\le \mathscr{A}^k(d(f^{i_0}, f^l x) + d(f^l, f^{j_0} x) - p) \\ &\le \mathscr{A}^k(u(x) - p) + \mathscr{A}^k(1 - \mathscr{A})^{-1}\mathscr{A}(u(x) - p) \\ &\le (1 - \mathscr{A})^{-1}\mathscr{A}(u(x) - p), \end{aligned}$$

since $j_0 \le j < n_0$, so the inequality (16) holds in this case.

Case IV: If $i = n_0$, $j \le l$, the triangular inequality, $\mathscr{A}(P) \subseteq P$ and (18) imply

$$\begin{aligned} d(f^{n_0}, f^l x) &\le \mathscr{A}(d(f^{n_0} x, f^j x) - p) \\ &\le \mathscr{A}(d(f^{n_0}, f^l x) + d(f^l, f^j x) - p) \\ &\le \mathscr{A}(d(f^{n_0}, f^l x) + \mathscr{A}(u(x) - p), \end{aligned}$$

$$d(f^{n_0}, f^l x) \le (1 - \mathscr{A})^{-1}\mathscr{A}(u(x) - p), \tag{20}$$

so inequality (16) is satisfied.

Case V: If $i = n_0$, $l < j \le n_0$. If $j = n_0$, it follows $d(f^{n_0}, f^l x) \le \mathscr{A}(-p)$. In either case,

$$d(f^{n_0}(x), f^l(x)) \le \mathscr{A}(d(f^j x, f^{n_0} x) - p) \tag{21}$$

and there exists $i_0 \le j_0 \le n_0$, $i_0 < l$ and some $1 < k_0$, such that

$$d(f^j(x), f^{n_0}(x)) \le \mathscr{A}^{k_0}(d(f^{i_0} x, f^{j_0} x)). \tag{22}$$

If $j_0 \le l$, then (16) follows by the last inequality and (21). If $j_0 < n_0$, then

$$\begin{aligned} d(f^{n_0}, f^l x) &\le \mathscr{A}^{1+k_0}(d(f^{i_0} x, f^{j_0} x) - p) \\ &\le \mathscr{A}^{1+k_0}(d(f^{i_0}, f^l x) + d(f^l, f^{j_0} x) - p) \\ &= \mathscr{A}^{1+k_0}(d(f^{i_0}, f^l x) - p) + \mathscr{A}^{1+k_0}(d(f^l, f^{j_0} x)) \end{aligned}$$

$$\begin{aligned}
&\leq \mathscr{A}^{1+k_0}(u(x)-p)+\mathscr{A}^{1+k_0}(1-\mathscr{A})^{-1}\mathscr{A}(u(x)-p)\\
&\leq \mathscr{A}^{1+k_0}(1-\mathscr{A})^{-1}(1-\mathscr{A}+\mathscr{A})(u(x)-p)\\
&= \mathscr{A}^{k_0}(1-\mathscr{A})^{-1}\mathscr{A}(u(x)-p)\\
&\leq (1-\mathscr{A})^{-1}\mathscr{A}(u(x)-p). \qquad (23)
\end{aligned}$$

But if $j_0 = n_0$, then

$$d(f^{n_0}, f^l x) \leq \mathscr{A}^{1+k_0}(d(f^{i_0}, f^l x)-p)+\mathscr{A}^{1+k_0}(d(f^l, f^{n_0}x)). \qquad (24)$$

Then, for some $k_1 \geq 1$ and $i_1 \leq j_1 \leq n_0$, $i_1 < l$, $d(f^l, f^{n_0}x) \leq \mathscr{A}^{k_1}(d(f^l x, f^{n_0}x)$, so by (24), we get

$$d(f^{n_0}, f^l x) \leq \mathscr{A}^{1+k_0}(d(f^{i_0}, f^l x)-p)+\mathscr{A}^{1+k_0+k_1}(d(f^{i_1}, f^{j_1}x)). \qquad (25)$$

Again, if $j_1 < n_0$, as in (23), we have

$$d(f^n(x), f^l(x)) \leq \mathscr{A}(1-\mathscr{A})^{-1}(u(x)-p), \quad n \geq l. \qquad (26)$$

In either case

$$d(f^{n_0}, f^l x) \leq \mathscr{A}^{1+k_0}(d(f^{i_0}, f^l x)-p)+\mathscr{A}^{1+k_0+k_1}(d(f^{i_1}, f^l x))+\mathscr{A}^{1+k_0+2k_1}(d(f^{i_1}, f^{n_0}x)). \qquad (27)$$

Hence, for arbitrary $n \in \mathbb{N}$

$$\begin{aligned}
d(f^{n_0}, f^l x) &\leq \mathscr{A}^{1+k_0}(d(f^{i_0}, f^l x)-p)+\sum_{m=1}^{n-1}\mathscr{A}^{1+k_0+mk_1}(d(f^{i_1}, f^l x))+\mathscr{A}^{1+k_0+nk_1}(d(f^{i_1}, f^{n_0}x))\\
&\leq \sum_{m=0}^{n-1}\mathscr{A}^{1+k_0+mk_1}\mathscr{A}(u(x)-p)+\mathscr{A}^{1+k_0+nk_1}(d(f^{i_1}, f^{n_0}x))\\
&\leq (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}^{1+k_0}(u(x)-p)+\mathscr{A}^{1+k_0+nk_1}(d(f^{i_1}x, f^{n_0}x))\\
&\leq (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(u(x)-p)+\mathscr{A}^{1+k_0+nk_1}(d(f^{i_1}x, f^{n_0}x)).
\end{aligned}$$

However, $\mathscr{A}^{1+k_0+nk_1}(d(f^{i_1}x, f^{n_0}x)) \to 0$ as $n \to \infty$. For each $c \gg 0$ there exists $n_c \in \mathbb{N}$ such that $\mathscr{A}^{1+k_0+nk_1}(d(f^{i_1}, f^{n_0}x)) \gg c$ for $n > n_c$, so

$$d(f^{n_0}x, f^l x) \leq (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(u(x)-p)+c, \quad c \gg 0,$$

i.e. $\quad d(f^{n_0}, f^l x) \leq (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(u(x)-p)$.

Thus by induction we obtained (16) for every $n \in \mathbb{N}$. Now, we prove that for each n

$$d(f^n x, f^j x) \leq (\mathscr{I}-\mathscr{A})^{-1}(u(x)-p), \quad j = 0, 1, 2, \cdots, l. \qquad (28)$$

This follows by (16), since

$$\begin{aligned}
d(f^n x, f^j x) &\leq d(f^n x, f^l x)+d(f^l x, f^j x)\\
&\leq (\mathscr{I}-\mathscr{A})^{-1}\mathscr{A}(u(x)-p)+u(x)\\
&= (\mathscr{I}-\mathscr{A})^{-1}(u(x)-p).
\end{aligned}$$

Since f is a Fisher -Perov quasi contraction, and by (28) we get that for $n > m \geq l$, $m = kl + r$, $0 \leq r < l$, $k \geq 1$

$$d(f^n x, f^m x) \leq \mathscr{A}^k(d(f^i x, f^j x)) \leq \mathscr{A}^k(\mathscr{I}-\mathscr{A})^{-1}(u(x)-p),$$

where $0 \leq i \leq j$ and $i \leq l$.

Since $\mathscr{A}^k(\mathscr{I}-\mathscr{A})^{-1}(u(x)-p) \to 0$, $k \to \infty (m \to \infty)$, implies that $\{f^n x\}$ is a Cauchy sequence in A and there exists $x^* \in A$ such that $\lim_{n\to\infty} f^n(x) = x^*$..

We will prove that f has a best proximity point x^* in A. Suppose that $c \ll 0$ and $\epsilon \gg 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$d(x^*, f^n(x)) \ll c,\ d(f^n(x), f^m(x)) \ll \epsilon \ \text{ and } \ d(x^*, f^n(x)) \ll \epsilon \ \text{ for all } \ n, m \geq n_0. \qquad (29)$$

Now, for each $n > n_0$,

$$d(x^*, f(x^*)) - p \ll d(x^*, f^n(x)) + d(f^n(x), f(x^*)) - p \le c + d(f(x^*), f^n(x)) - p. \tag{30}$$

Furthermore, because f is a Fisher-Perov quasi contraction, we have

$$d(f^n(x), f(x^*)) \le \mathscr{A}(u(x)) \tag{31}$$

for some

$$u(x) \in \{d(f^r(x), fx^*) - p, d(f^r(x), x^*) - p, d(f^r(x), f^{r'}(x)) - p, d(x^*, fx^*) - p : 0 \le r, r' \le n\}.$$

But

$$d(f^r, fx^*) - p \le d(f^r, x^*) + d(x^*, fx^*) - p,$$

since $f^n x \to x^*$ as $n \to \infty$, so for each $c \gg 0$ we may choose n_0 for which $d(f^r x, x^*), d(f^n x, f^m x) \ll c$, $n, m \ge n_0$. Choose $n > n_0 + p$, then

$$d(x^*, fx^*) \le c + \mathscr{A}(d(x^*, fx^*)) + \mathscr{A}(c)$$

for any $c \gg 0$. For $c = \frac{c}{n}$, $n \in \mathbb{N}$, we get $d(x^*, fx^*) - p \le \mathscr{A}(d(x^*, fx^*) - p)$. Since $(\mathscr{I} - \mathscr{A})^{-1}(P) \subseteq P$, we have $d(x^*, fx^*) = p$. □

If we take $q = 1$ (or $p = 1$), then we can omit the condition of continuity in Theorem 4.2. Then, we have:

Theorem 4.3. *Let (X, d) be a complete cone metric space, $\operatorname{int}(P) \neq \phi$. A and B closed subsets of X. Let $f : A \to B$ be continuous $(l, 1)$-Perov quasi contraction, $\mathscr{A}(P) \subseteq P$ and $f(A) \subseteq B$. Then f has a unique best proximity point x^* in A.*

Proof. Let $x \in X$. Then by Theorem 4.2, $\{f^n x\}$ is a Cauchy sequence in A. Closedness of A implies $f^n x \to x^*$ as $n \to \infty$. For $n > l$, we have

$$\begin{aligned} d(x^*, fx^*) - p &\le d(x^*, f^n x) + d(f^n x, fx^*) - p \\ &= d(x^*, f^n x) + d(f^r f^{n-r} x, fx^*) - p \\ &\le d(x^*, f^n x) + \mathscr{A}(u(x)), \end{aligned}$$

where

$$u(x) \in \{d(f^r f^{n-l} x, fx^*) - p, d(f^r f^{n-l} x, x^*) - p, d(f^r f^{n-l} x, f^{r'} f^{n-l} x^*) - p, d(x^*, fx^*) - p : 0 \le r, r' \le l\}.$$

But

$$d(f^r f^{n-l} x, fx^*) - p \le d(f^r f^{n-l} x, x^*) + d(x^*, fx^*) - p,$$

since $f^n x \to x^*$ as $n \to \infty$, so for each $c \gg 0$ we may choose n_0 for which $d(f^n x, x^*), d(f^n x, f^m x) \ll$, $n, m \ge n_0$. Choose $n > n_0 + p$, then

$$d(x^*, fx^*) \le c + \mathscr{A}(d(x^*, fx^*)) + \mathscr{A}(c)$$

for any $c \gg 0$. For $c = \frac{c}{n}$, $n \in \mathbb{N}$, we get $d(x^*, fx^*) - p \le \mathscr{A}(d(x^*, fx^*) - p)$. Since $(\mathscr{I} - \mathscr{A})^{-1}(P) \subseteq P$, we have $d(x^*, fx^*) = p$. □

Remark 4.4. By taking $l = m = 1$ in Theorem 4.2, we obtain Theorem 3.2.

5. Conclusion

This paper concerned with the existence of best proximity point theorems for Ćirić-Perov quasi contraction and Fisher-Perov quasi contraction with non-normal cone. An example is given to support our result. The presented results generalized the main results of M. Cvetković and V. Rakočević [9, 10]

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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