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Research Article

A Modified Subgradient Extragradient Algorithm with Inertial Effects

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Abstract. In this article, we introduce an inertial modified subgradient extragradient method by combining inertial type algorithm with modified subgradient extragradient method and for solving the *variational inequality* (VI) in a Hilbert space H. Also, we establish a weak convergence theorem for proposed algorithm. Finally, we describe the performance of our proposed algorithm with the help of numerical experiment and we show the efficiency and advantage of the inertial modified subgradient extragradient method.

Keywords. Variational inequality; Inertial type algorithm; Extragradient method; Subgradient extragradient method; Projection and contraction method

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1. Introduction

In this article, we will focus on finding the solution of classical *Variational Inequality* (VI) problem, that is to find u in non-empty, closed, convex set C in Hilbert space H with inner product $\langle\cdot\rangle$ such that

$$\langle fu, x-u\rangle \geq 0, \tag{1.1}$$

for all $x \in C$ where $f: H \to H$ is defined mapping. This problem captures various applications arising in many areas, such as partial differential equations, optimal control, optimization, mathematical programming and some other nonlinear problems (see, e.g., [19] and references therein). It is well-known that if f is L-Lipschitz continuous and η-strongly monotone on C, that is,

$$\|f(x)-f(y)\| \leq L\|x-y\|, \quad \forall\, x,y \in C,$$

and

$$\langle f(x)-f(y), x-y\rangle \geq \eta\|x-y\|^2, \quad \forall\, x,y \in C,$$

where $L>0$ and $\eta>0$ are the Lipschitz and strong monotonicity constant, respectively, then the variational inequality (1.1) has a unique solution.

However, if f is simply L-Lipschitz continuous and monotone on C, that is,

$$\langle f(x)-f(y), x-y\rangle \geq 0, \quad \forall\, x,y \in C,$$

but not η-strongly monotone, then the variational inequality (1.1) may fails to get a solution.

Several authors have introduced and analyzed several iterative methods for solving the variational inequality (1.1). The one step projection method for optimization problems:

$$x_{k+1} = P_C(x_k - \alpha_k f(x_k)),$$

for each $k \geq 1$, where $\alpha_k \in \left(0, \frac{2\eta}{L^2}\right)$ and P_C denotes the Euclidean least distance projection onto C. The projection method provided that the mapping f is L-Lipschitz continuous and η-strongly monotone. But if the strong monotonicity is relaxed to plain monotonicity then the projected gradient method may diverge (see [24] for counter example).

To avoid the assumption of strong monotonicity, Korpelevich [16] proposed the extragradient method:

$$\begin{cases} y_k = P_C = (x_k - \alpha_k f(x_k)) \\ x_{k+1} = P_C(x_k - \alpha_k f(y_k)) \end{cases}$$

for each $k \geq 1$, which converges if f is Lipschitz and monotone, where $\alpha_k \in \left(0, \frac{1}{L}\right)$ and L is the Lipschitz constant of f, or α_k is updated by the following adaptive procedure as:

$$\alpha_k\|f(x_k)-f(y_k)\| \leq \mu\|x_k - y_k\|, \tag{1.2}$$

where $\mu \in (0,1)$.

The extragradient method has received a considerable attention and many authors modified and improved it in various ways; there is one famous extension of He [12] and Sun [22], called the projection and contraction method.

Algorithm 1.1 (The projection and contraction method)**.**

$$\begin{cases} y_k = P_C(x_k - \alpha_k f(x_k)) \\ x_{k+1} = P_C(x_k - \gamma\rho_k\alpha_k f(y_k)) \end{cases}$$

where $\gamma \in (0,2)$, $\alpha_k \in \left(0, \frac{1}{L}\right)$ or $\{\alpha_k\}_{k=0}^{\infty}$ is selected self-adaptively, and

$$\rho_k = \frac{\|x_k - y_k\|^2 - \alpha_k\langle x_k - y_k, (f(x_k), f(y_k))\rangle}{\|(x_k - y_k) - \alpha_k(f(x_k) - f(y_k))\|^2}.$$

The choice of the stepsize is very important since the performance of the iteration methods heavily depends on it. We can easily noticed that in the classical extragradient method, the stepsize α_k is same in both projections but in Projection and contraction method, stepsizes are different in both projections. The numerical example proved shows that the computational load of Extragradient method is about double of that projection and contraction method.

In fact, it is seen that, in the extragradient method one needs to calculate two orthogonal projections onto C in each iteration, where projection onto a closed, convex set C is related to a minimum distance problem.

To overcome this obstacle, Censor *et al.* in [3] introduced the subgradient extragradient method in which the second projection onto the C is replaced by a specific subgradient extragradient projection which can be easily calculated.

Algorithm 1.2 (The subgradient extragradient method)**.**

$$\begin{cases} y_k = P_C(x_k - \alpha_k f(x_k)) \\ x_{k+1} = P_{T_k}(x_k - \alpha_k f(y_k)) \end{cases}$$

where T_k is the set defined as

$$T_k := \{w \in H : \langle (x_k - \alpha_k f(x_k)) - y_k, w - y_k\rangle \le 0\},$$

and $\alpha_k \in \left(0, \frac{1}{L}\right)$ or $\{\alpha_k\}_{k=0}^{\infty}$ is selected self-adaptively, that is, $\alpha_k = \sigma\rho_{m_k}$, $\alpha > 0$, $\rho \in (0,1)$ and m_k is the smallest non-negative integer such that

$$\alpha_k\|f(x_k - f(y_k))\| \le \mu\|x_k - y_k\|, \quad \mu \in (0,1). \tag{1.3}$$

Since the inception of the subgradient extragradient method, many authors have proposed various modifications. So, the stepsizes used in extragradient and subgradient extragradient methods has an essential role in the convergence rate of the two step methods, hence to modify the stepsize in the second step of the subgradient extragradient method in the sense of He [12] and Sun [22], Dong *et al*. [4] introduced a modified subgradient extragradient method which improves the step size in the second step of the subgradient extragradient method.

Algorithm 1.3 (The modified subgradient extragradient method)**.**

Step 0: Select a starting point $x_0 \in H$ and set $k = 0$.

Step 1: Given the current iterate x_k, compute

$$\alpha_k\|f(x_k) - f(y_k)\| \le \mu\|x_k - y_k\|, \quad \mu \in (0,1). \tag{1.4}$$

Step 2: Construct the set

$$T_k := \{w \in H : \langle (x_k - \alpha_k f(x_k)) - y_k, w - y_k\rangle \le 0\}, \tag{1.5}$$

$$x_{k+1} = P_{T_k}(x_k - \gamma\rho_k\alpha_k f(y_k)), \tag{1.6}$$

where $\gamma \in (0,2)$ and

$$\rho_k := \frac{\langle (x_k - y_k, d(x_k), y_k) \rangle}{\|d(x_k, y_k)\|^2}, \tag{1.7}$$

where

$$d(x_k, y_k) = (x_k - y_k) - \alpha_k(f(x_k) - f(y_k)). \tag{1.8}$$

Set $k \leftarrow k+1$ and return to *Step* 1.

Question. Is it possible to obtain the solution of *Variational Inequality* (VI) with the help of less number of iterations, less number of total iteration to find suitable α_k, and within shortest time, in the spirit of Dong *et al.* [4].

In this paper, we provide an affirmative answer to this question by relying on the work of [4] and introduce a modified subgradient extragradient method with inertial effects. The convergence of the proposed method is proved under standard assumptions and numerical experiment validates its applicability.

The next sections are arranged as follows: In Section 2, we recalled some basic notions and results which will use in sequel. In Section 3, we presented and analyzed an inertial modified subgradient extragradient method. In Section 4, we give numerical analysis to illustrate the efficiency and advantage of the inertial modified subgradient extragradient method and compared the performance of our proposed method with other methods numerically and graphically. In Section 5, we have added concluding remarks.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$ and let D be nonempty, closed and convex subset of H. For given a sequence $\{x_k\}$ with subsequence $\{x_{k_j}\}$, we will use the following notations in sequel:

(1) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;

(2) $\omega_w(x_k) = \{x : \exists\, x_{k_j} \rightharpoonup x\}$ denoted the weak ω-limit set of $\{x_k\}$.

For real Hilbert space H with inner product $\langle\cdot\rangle$ and the induced norm $\|\cdot\|$, and let D be a nonempty, closed and convex subset of H. We need some results and tools which are listed below. Recall that, in a Hilbert space H,

$$\|\eta x + (1-\eta)y\|^2 = \eta\|x\|^2 + (1-\eta)\|y\|^2 - \eta(1-\eta)\|x-y\|^2, \tag{2.1}$$

for all $x, y \in H$ and $\eta \in \mathbb{R}$ (see in Corollary [1]). **Corollary number missing**

For each point $x \in H$, there exists a unique nearest point in D, denoted by $P_D(x)$, i.e.,

$$\|x - P_D(x)\| \le \|x - y\|.$$

Definition 2.1. Let $B : H \rightrightarrows 2^H$ be a point to set operator defined on a real Hilbert space H. B is caller a maximal monotone operator if B is monotone, i.e.,

$$\langle u - v, x - y\rangle \ge 0,$$

for all $u \in B(x)$ and $v \in B(y)$ and the graph $G(B)$ of B,

$$G(B) := \{(x,u) \in H \times H : u \in B(x)\},$$

is not properly contained in the graph of any other monotone operator.

It is obvious that a monotone mapping B is maximal if and only if, for any $(x,u) \in H \times H$, if $\langle u-v, x-y\rangle \geq 0$ for all $(v,y) \in G(B)$, then it follows that $u \in B(x)$.

Definition 2.2. The normal cone of D at $v \in D$, denoted by $N_D(v)$, is defined as

$$N_D(v) := \{d \in H : \langle d, y-v\rangle\} \leq 0,$$

for all $y \in D$.

Lemma 2.1 ([8])**.** *Let $x \in H$ and $z \in D$. Then, $z = P_D(x)$ if and only if*

$$P_D(x) \in D \tag{2.2}$$

and

$$\langle x - P_D(x), P_D(x) - y\rangle \geq 0, \quad \textit{for all } x \in H, y \in D. \tag{2.3}$$

Lemma 2.2 ([8])**.** *For any $x, y \in H$ and $z \in D$, it holds:*

1. $\|P_D(x) - P_D(y)\| \leq \|x-y\|$;
2. $\|P_D(x) - z\| \leq \|x-z\|^2 - \|P_D(x) - x\|^2$.

Lemma 2.3 ([1])**.** *Let C be a non empty set of H and $\{x_k\}$ be a sequence in H such that the following two conditions hold:*

(i) *for all $x \in C$, $\lim\limits_{k\to\infty} \|x_k - x\|$ exists;*

(ii) *every sequential weak cluster point of $\{x_k\}$ is in C.*

Then, the sequence $\{x_k\}$ converges weakly to a point in C.

3. Main Results

In this section, we present the inertial modified subgradient extragradient algorithm and examine its convergence. Consider the mapping $f : H \to H$, we introduce the following algorithm:

Algorithm 3.1 (The modified subgradient extragradient method with inertial effects)**.** Take $\sigma > 0$, $\rho > 0$ and $\mu \in (0,1)$.

Step 0: Choose initial guesses $x_0, x_1 \in H$ arbitrarily.
Calculate the $(k+1)$th iterate x_{k+1} via the formula:

Step 1: Under current iterate, compute w_k

$$w_k = x_k + \alpha_k(x_k - x_{k-1}). \tag{3.1}$$

Step 2: Calculate

$$y_k = P_C(w_k - \alpha_k F(w_k)), \tag{3.2}$$

$$\alpha_k \|F(w_k) - F(y_k)\| \leq \mu \|w_k - y_k\|, \tag{3.3}$$

Step 3: Construct the set

$$T_k := \{u \in H : \langle (w_k - \alpha_k F(w_k)) - y_k, u - y_k\rangle \le 0\}, \tag{3.4}$$

$$x_{k+1} = P_{T_k}(w_k - \gamma\rho_k\alpha_k F(y_k)) \tag{3.5}$$

for each $k \ge 1$, where $\mu, \rho \in (0,1)$, $\sigma > 0$ and $\gamma \in (0,2)$ and

$$\rho_k := \frac{\langle w_k - y_k, d(w_k, y_k)\rangle}{\|d(w_k, y_k)\|^2}, \tag{3.6}$$

with

$$d(w_k, y_k) = (w_k - y_k) - \alpha_k(F(w_k) - F(y_k)), \tag{3.7}$$

and $\{\alpha_k\}$ is selected self adaptive, that is, $\alpha_k = \sigma\rho^{m_k}$, $\alpha > 0$, and m_k is the smallest non-negative integer which satisfies inequality with $k \ge 1$.

If $y_k = w_k$ or $d(w_k, y_k) = 0$ then x_{k+1} is the solution of the variational inequality (1.1) and iterative process stops; otherwise we will continue and obtain the next iterate x_{k+2} from our defined Algorithm 3.1.

3.1 Convergence Analysis

In this subsection, we will examine the convergence of our defined Algorithm 3.1. We establish a sequence $\{w_k\}$ which is obtained by Algorithm 3.1, is weakly convergent to the solution of variational inequality (1.1). We construct the weak convergence theorem and its proof for inertial modified subgradient extragradient Algorithm 3.1.

To discuss the convergence of the Algorithm 3.1, suppose that following conditions hold:

Condition 3.1.1. *The solution set of* (1.1)*, denoted by SOL*(C, f)*, is nonempty.*

Condition 3.1.2. *The mapping f is monotone on H, i.e.,*

$$\langle f(x) - f(y), x - y\rangle \ge 0,$$

for all $x, y \in H$.

Condition 3.1.3. *The mapping f is Lipschitz continuous on H with Lipscchitz constant $L > 0$, i.e.,*

$$\|f(x) - f(y)\| \le L\|x - y\|,$$

for all $x, y \in H$.

Consider the following assumptions on our $\{\alpha_k\}$:

Condition 3.1.4. $0 \le \alpha_k \le \alpha$.

Condition 3.1.5. $\sum_{k=1}^{+\infty} \alpha_k \|x_k - x_{k-1}\|^2 < +\infty$.

Condition 3.1.6. $\lim_{k\to\infty} \alpha_k \|x_k - y_k\| = 0$.

Lemma 3.1 ([5]). *Let K be closed, convex subset of real Hilbert space H and P_K be the (metric or nearest point) projection from H onto K (that is, for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$). Then, for any $x \in H$ and $z \in K$, $z = P_K x$ if and only if there holds the relation:*

$$\langle x - z, y - z \rangle \le 0,$$

for all $y \in K$.

Lemma 3.2 ([4]). *Let $\{\rho_k\}$ be a sequence defined by (3.17) in our algorithm. Then, under Condition 3.1.2 and 3.1.3, we have*

$$\rho_k \ge \frac{1-\mu}{1+\mu^2}.$$

Lemma 3.3. *Suppose that $0 < \alpha_k \le \alpha < \frac{1}{L}$. Let $y_k = w_k$ or $d(w_k, y_k) = 0$ in algorithm, then $x_{k+1} \in SOL(C, f)$.*

Proof. From Conditions 3.1.3 and Algorithm 3.1, it follows

$$\begin{aligned}\|d(w_k, y_k)\| &= \|(w_k - y_k) - \alpha_k (f(w_k) - f(y_k))\| \\ &\ge \|w_k - y_k\| - \alpha_k \|f(w_k) - f(y_k)\| \\ &\ge (1 - \alpha_k L)\|w_k - y_k\|.\end{aligned}$$

Similarly, we can show that

$$\|d(w_k, y_k)\| \le (1 + \alpha_k L)\|w_k - y_k\|.$$

So, $d(w_k, y_k) = 0$ if and only if $y_k = w_k$. From our defined algorithm, we have

$$y_k = P_C(y_k - \alpha_k f(y_k)).$$

When $d(w_k, y_k) = 0$ then one can obtain that $\rho_k = 0$, and

$$P_{T_k}(y_k - 0) = y_k,$$

and so $x_{k+1} = y_k$, where

$$y_k = P_C(y_k - \alpha_k f(y_k)),$$

which with Lemma 2.2, yields that $x_{k+1} \in SOL(C, f)$. Hence completes the proof. □

In above lemma, we see that if our defined algorithm terminates in a finite step of iterations, then x_k is the solution of the variational inequality (1.1). So, in the rest of this section, we assume that our algorithm does not terminate in any finite iterations, and generates an infinite sequence.

Lemma 3.4. *Let $\{x_k\}_0^\infty$ be a sequence generated by our algorithm and let $u \in SOL(C, f)$. Then, under Condition 3.1.1, 3.1.2 and 3.1.3, we have the following:*

$$\|x_{k+1} - u\|^2 \le \|w_k - u\|^2 - \|(w_k - x_{k+1}) - \gamma \rho_k d(w_k, y_k)\|^2 - \gamma(2-\gamma)\rho_k^2 \|d(w_k, y_k)\|^2.$$

Proof. By definition of x_{k+1} and by Lemma 2.1, we have

$$\begin{aligned}\|x_{k+1} - u\|^2 &\le \|w_k - \gamma \rho_k \alpha_k f(y_k) - u\|^2 - \|w_k \gamma \rho_k \alpha_k f(y_k) - x_{k+1}\|^2 \\ &= \|w_k - u\|^2 - \|x_{k+1} - w_k\|^2 - 2\gamma \rho_k \alpha_k \langle x_{k+1} - u, f(y_k)\rangle.\end{aligned}$$

Since, $u \in SOL(C,f)$ and f is monotone, we have

$$\langle f(y_k) - f(u), y_k - u\rangle \geq 0,$$

which with (1.1) implies

$$\langle f(y_k), y_k - u\rangle \geq 0,$$

for all $k \geq 0$. So,

$$\langle f(y_k), x_{k+1} - u\rangle \geq \langle f(y_k), x_{k+1} - y_k\rangle.$$

By the definition of T_k and $x_{k+1} \in T_k$, we have

$$\langle (w_k) - \alpha_k f(w_k) - y_k, x_{k+1} - y_k\rangle \leq 0,$$

which implies

$$\langle d(w_k, y_k), x_{k+1} - y_k\rangle \leq \alpha_k \langle f(y_k), x_{k+1} - y_k\rangle.$$

Using above two inequalities, we get

$$\begin{aligned}
-2\gamma\rho_k\alpha_k\langle x_{k+1} - u, f(y_k)\rangle &\leq -2\gamma\rho_k\langle x_{k+1} - y_k, d(w_k, y_k)\rangle \\
&= -2\gamma\rho_k\langle w_k - y_k, d(w_k, y_k)\rangle + 2\gamma\rho_k\langle w_k - x_{k+1}, d(w_k, y_k)\rangle \\
&= -2\gamma\rho_k^2\|d(w_k, y_k)\|^2 + 2\gamma\rho_k\langle w_k - x_{k+1}, d(w_k, y_k)\rangle \\
&= -2\gamma\rho_k^2\|d(w_k, y_k)\|^2 - \|(w_k - x_{k+1}) - \gamma\rho_k d(w_k, y_k)\|^2 \\
&\quad + \|w_k - x_{k+1}\|^2 + \gamma^2\rho_k^2\|d(w_k, y_k)\|^2.
\end{aligned}$$

By putting this value in the first inequality of proof, we get as

$$\|x_{k+1} - u\|^2 \leq \|w_k - u\|^2 - \|(w_k - x_{k+1}) - \gamma\rho_k d(w_k, y_k)\|^2 - \gamma(2-\gamma)\rho_k^2\|d(w_k, y_k)\|^2. \qquad \square$$

Theorem 3.1. *Assume the Condition* 3.1.1-3.1.6 *hold. Then, the sequence* $\{x_k\}$ *generated by Algorithm* 3.1 *converges weakly to a solution of the variational inequality problem* (1.1).

Proof. Fix $u \in SOL(C,f)$. Applying (2.1), we have

$$\begin{aligned}
\|w_k - u\|^2 &= \|x_k + \alpha_k(x_k - x_{k-1}) - u\|^2 \\
&= \|x_k + \alpha_k(x_k - x_{k-1}) + \alpha_k u - \alpha_k u - u\|^2 \\
&= \|(1+\alpha_k)(x_k - u) - \alpha_k(x_{k-1} - u)\|^2 \\
&= (1+\alpha_k)\|x_k - u\|^2 - \alpha_k\|x_{k-1} - u\|^2 + \alpha_k(1+\alpha_k)\|x_k - x_{k-1}\|^2.
\end{aligned} \tag{3.8}$$

Hence, from Lemma 3.4 and (3.8), it follows that,

$$\begin{aligned}
\|x_{k+1} - u\|^2 &\leq \|w_k - u\|^2 - \|(w_k - x_{k+1}) - \gamma\rho_k d(w_k, y_k)\|^2 - \frac{\gamma(2-\gamma)(1-\mu)^2}{1+\mu^2}\|w_k - y_k\|^2 \\
&= [(1+\alpha_k)\|x_k - u\|^2 - \alpha_k\|x_{k-1} - u\|^2 + \alpha_k(1+\alpha_k)\|x_k - x_{k-1}\|^2] \\
&\quad - \|(w_k - x_{k+1}) - \gamma\rho_k d(w_k, y_k)\|^2 - \frac{\gamma(2-\gamma)(1-\mu)^2}{1+\mu^2}\|w_k - y_k\|^2 \\
&\leq [(1+\alpha_k)\|x_k - u\|^2 - \alpha_k\|x_{k-1} - u\|^2 + \alpha_k(1+\alpha_k)\|x_k - x_{k-1}\|^2] - \|x_{k+1} - w_k\|^2.
\end{aligned}$$

Hence, we can write as

$$\|x_{k+1} - u\|^2 - (1+\alpha_k)\|x_k - u\|^2 + \alpha_k\|x_{k-1} - u\|^2 \leq \alpha_k(1+\alpha_k)\|x_k - x_{k-1}\|^2 - \|x_{k+1} - w_k\|^2. \tag{3.9}$$

Now, we have to calculate the value of $\|x_{k+1}-w_k\|^2$, as follows:

$$\begin{aligned}\|x_{k+1}-w_k\|^2 &= \|(x_{k+1}-x_k)-\alpha_k(x_k-x_{k-1})\|^2\\ &= \|x_{k+1}-x_k\|^2+\alpha_k\|x_k-x_{k-1}\|^2-2\alpha_k\langle x_{k+1}-x_k,x_k-x_{k-1}\rangle\\ &\le \|(x_{k+1}-x_k\|^2+\alpha_k\|x_k-x_{k-1}\|^2.\end{aligned} \tag{3.10}$$

Put this value (3.10) in (3.9), we get

$$\|x_{k+1}-u\|^2-(1+\alpha_k)\|x_k-u\|^2+\alpha_k\|x_{k-1}-u\|^2\le\alpha_k\|x_k-x_{k-1}\|^2-\|x_{k+1}-x_k\|^2. \tag{3.11}$$

This implies that

$$\begin{aligned}\|x_{k+1}-u\|^2 &\le (1+\alpha_k)\|x_k-u\|^2-\alpha_k\|x_{k-1}-u\|^2+\alpha_k\|x_k-x_{k-1}\|^2-\|x_{k+1}-x_k\|^2\\ &\le (1+\alpha_k)\|x_k-u\|^2+\alpha_k\|x_k-x_{k-1}\|^2.\end{aligned} \tag{3.12}$$

Since by Condition 3.1.6, we may say, $\lim_{k\to\infty}\|x_{k+1}-u\|^2\le\lim_{k\to\infty}(1+\alpha_k)\|x_k-u\|^2$ which implies that the sequence $\{x_{k+1}-u\}$ is decreasing sequence and bounded below thus converges to some finite limit. Also, $\{x_k\}$ is Fejer Monotone with respect to $SOL(C,f)$ and thus is bounded, so it seems the existence of $\lim_{k\to\infty}\|x_k-u\|^2$, it follows that

$$\sum_{k=0}^{\infty}\|x_k-y_k\|^2\le+\infty \tag{3.13}$$

which implies $\lim_{n\to\infty}\|x_k-y_k\|=0$. Then by (3.12), we can say

$$\begin{aligned}\lim_{k\to\infty}\|x_{k+1}-x_k\| &\le \lim_{k\to\infty}\|x_{k+1}-y_k\|+\lim_{k\to\infty}\|y_k-x_k\|\\ &\le \lim_{k\to\infty}(1+\alpha_k)\|x_k-y_k\|\\ &=0,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\lim_{k\to\infty}\|x_{k+1}-w_k\| &\le \lim_{k\to\infty}\|x_{k+1}-x_k\|+\alpha_k\lim_{k\to\infty}\|x_k-x_{k-1}\|\\ &\le \lim_{k\to\infty}\|x_{k+1}-x_k\|+\alpha\lim_{k\to\infty}\|x_k-x_{k-1}\|\\ &=0,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\lim_{k\to\infty}\|w_k-y_k\| &= \lim_{k\to\infty}\|x_k+\alpha_k(x_k-x_{k-1})-y_k\|^2\\ &= \lim_{k\to\infty}[\|x_k-y_k\|^2+\alpha_k^2\|x_k-x_{k-1}\|^2+2\alpha_k\langle x_k-y_k,x_k-x_{k-1}\rangle]\\ &=0.\end{aligned} \tag{3.16}$$

Now, we are to show that $\omega_w(x_k)\subset SOL(C,f)$. Due to the boundedness of $\{x_k\}$ it has atleast one weak cluster point. Let $x^*\in\omega_w(x_k)$ then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ which converges weakly to x^*. Also, it follows that $\{w_{k_i}\}$ and $\{y_{k_i}\}$ converges weakly to x^*.

Finally, we show that, x^* is the solution of variational inequality (1.1). Let

$$Av=\begin{cases} f(v)+N_C(v) & v\in C\\ \phi & \text{otherwise,}\end{cases} \tag{3.17}$$

where $N_C(v)$ is normal cone of C at $v\in C$, that is,

$$N_C(v):=\{d\in H:\langle d,y-v\rangle\le 0,\ \forall\ y\in C\}.$$

It is known that A is maximal monotone operator and $A^{-1}(0)=SOL(C,f)$. If $(v,w)\in G(A)$, then we have $w-f(v)\in N_C(v)$ since $w\in A(v)=f(v)+N_C(v)$.

Thus, it follows that

$$\langle w - f(v), v - y\rangle \geq 0 \tag{3.18}$$

for all $y \in C$. Since $y_{k_i} \in C$, we have

$$\langle w - f(v), v - y_{k_i}\rangle \geq 0.$$

On the other hand, by definition of y_k and Lemma 3.3, it follows that

$$\langle w_k - \alpha_k f(w_k) - y_k, y_k - v\rangle \geq 0$$

and, consequently, $\left\langle \frac{y_k - w_k}{\alpha_k} + f(w_k), v - y_k\right\rangle \geq 0$. Hence, we have

$$\begin{aligned}
\langle w, v - y_{k_i}\rangle &\geq \langle f(v), v - y_{k_i}\rangle \\
&\geq \langle f(v), v - y_{k_i}\rangle - \left\langle \frac{y_{k_i} - w_{k_i}}{\alpha_{k_i}} + f(w_{k_i}), v - y_{k_i}\right\rangle \\
&= \langle f(v) - f(y_{k_i}), v - y_k\rangle + \langle f(y_{k_i} - f(w_{k_i})), v - y_{k_i}\rangle - \left\langle \frac{y_{k_i} - w_{k_i}}{\alpha_{k_i}}, v - y_{k_i}\right\rangle \\
&\geq \langle f(y_{k_i}) - f(w_{k_i}), v - y_{k_i}\rangle - \left\langle \frac{y_{k_i} - w_{k_i}}{\alpha_{k_i}}, v - y_{k_i}\right\rangle,
\end{aligned} \tag{3.19}$$

which implies $\langle w, v - y_{k_i}\rangle \geq \langle f(y_{k_i}) - f(w_{k_i}), v - y_{k_i}\rangle - \left\langle \frac{y_{k_i} - w_{k_i}}{\alpha_{k_i}}, v - y_{k_i}\right\rangle$. Taking a limit as $i \to \infty$ in the above inequality, we obtain

$$\langle w, v - x^*\rangle.$$

Since A is maximal monotone operator, it follows that $x^* \in A^{-1}(0) = SOL(C,f)$. This complete the proof. □

By considering $x_k = x_{k-1}$, in our defined Algorithm 3.1, then Algorithm 3.1 converts into Algorithm 1.3, then we get Theorem 3.1 of [4] as a corollary, as follows:

Corollary 3.1. *Assume the Condition* 3.1.1-3.1.3 *hold. Then, the sequence* $\{x_k\}$ *generated by Algorithm* 1.3 *converges weakly to a solution of the variational inequality problem* (1.1)*.*

Proof. By putting $x_k = x_{k-1}$ in our Algorithm 3.1, we obtain desired result. □

4. Numerical Experiments

In this section, we evaluate the performance of proposed Algorithm 3.1, and present a numerical example relative to the variational inequality. We compare an inertial modified subgradient extragradient method Algorithm 3.1 with Algorithm 1.1 (The projection and contraction method), Algorithm 1.2 (The subgradient extragradient method) and Algorithm 1.3 (The modified subgradient extragradient method).

Consider the linear operator $Fx := Ax + b$, which is taken from [4] and has been considered by many authors for numerical experiments, where

$$A = BB^T + C + D,$$

and B is an $n \times n$ matrix, C is an $n \times n$ skew-symmetric matrix, D is an $n \times n$ diagonal matrix, whose diagonal entries are non-negative (so A is positive semi-definite), and b is a vector in $\mathbb{R}^n$.

The feasible set $N \subset \mathbb{R}^n$ is closed and convex and defined as

$$N := \{x \in \mathbb{R}^n \mid Qx \le p\},$$

where Q is $l \times n$ matrix and p is a non-negative vector. It is clear that F is monotone and L-Lipschitz continuous with $L = \|A\|$ (hence uniformly continuous). For $b = 0$, the solution set $SOL(N,F) = \{0\}$. Just as in [4], we randomly choose the starting points $x_0, x_1 \in [0,1]^n$ in Algorithm 1.1, Algorithm 1.2, Algorithm 1.3 and in Algorithm 3.1. We choose the stopping criterion as $\|x_k\| \le \epsilon = 0.005$ and the parameters $\sigma = 0.01$, $\rho = 0.4$, $\mu = 0.85$, and $\gamma = 1.99$. The size $l = 100$ and $n = 5,10,20,30,40,50,60,70$ and 80. The matrix B, C, D and the vector p are generated randomly.

In Table 1, we denoted 'the number of iterations' with 'Iter.' and 'the number of total iterations of finding suitable α_k' with 'Intl.' and we can easily examine that our defined algorithm (Algorithm 3.1 is highly efficient with respect to the number of iterations, the number of total iterations of finding suitable α_k and the CPU time comparatively Algorithm 1.1, Algorithm 1.2 and Algorithm 1.3. The given Table 1 shows that Algorithm 3.1 is more efficient than Algorithm 1.1, Algorithm 1.2 and Algorithm 1.3 with respect to iter., intl. and CPU time. Also, in the concluding remarks we can see the graph with error comparison of Algorithm 1.1, Algorithm 1.2, Algorithm 1.3 and Algorithm 3.1 respectively, from where we can easily see the efficiency of our defined algorithm with least error.

Table 1. Convergence comparison of Algorithms 1.1, 1.2, 1.3 and 3.1

	Iter.				Inlt.				CPU in seconds			
n	Alg. 1.1	Alg. 1.2	Alg. 1.3	Alg. 3.1	Alg. 1.1	Alg. 1.2	Alg. 1.3	Alg. 3.1	Alg. 1.1	Alg. 1.2	Alg. 1.3	Alg. 3.1
5	529	1893	216	1	529	1893	216	1	0.9219	0.7656	0.7656	0
10	267	841	369	1	267	841	369	1	1.375	2.125	0.7344	0
20	303	656	522	2	334	667	824	4	1.5313	0.8906	2.3906	0
30	467	1457	555	2	858	2940	1030	6	2.7344	1.4375	1.6563	0.0469
40	647	2895	917	2	1545	8531	2452	6	7.5938	43.8906	3.6875	0.0781
50	1374	5747	1649	2	4190	22321	5052	8	7.8906	66.5313	3.0156	0.0313
60	2187	7933	1971	2	7610	31732	6983	8	10.0781	79.3906	5.5313	0.0313
70	2201	8387	1567	2	8178	32992	6073	8	7.4844	151.2344	61.9219	0.0469
80	3334	7729	5321	2	13575	31907	26356	10	11.8594	11.2656	27.2813	0.0313

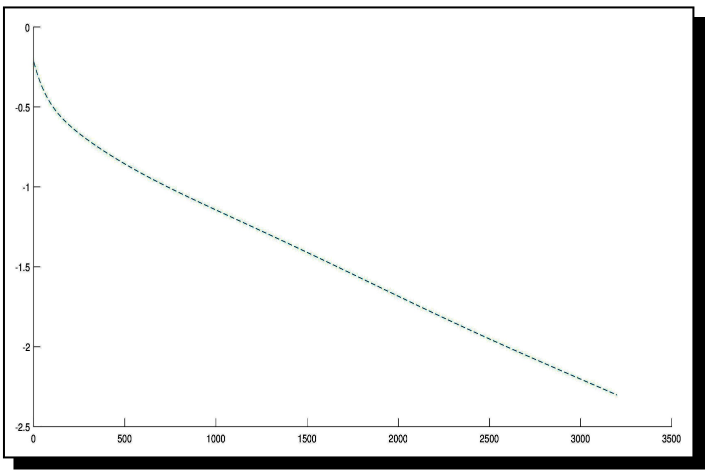

Figure 1. Algorithm 1.1

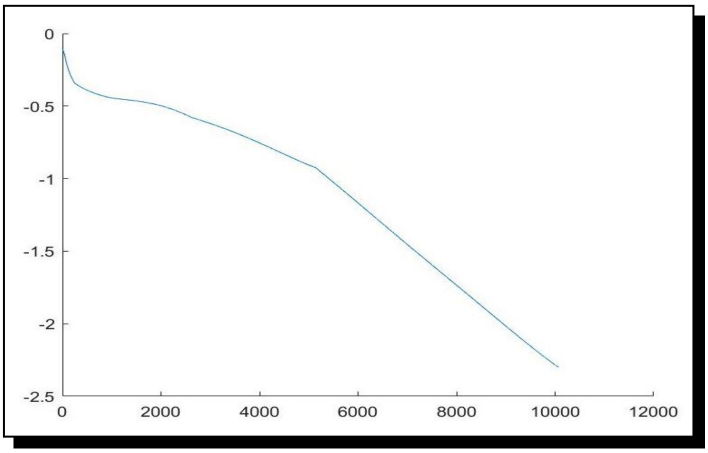

Figure 2. Algorithm 1.2

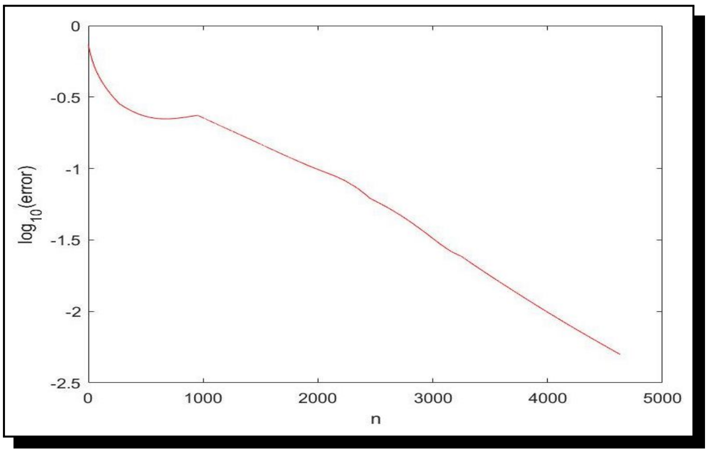

Figure 3. Algorithm 1.3

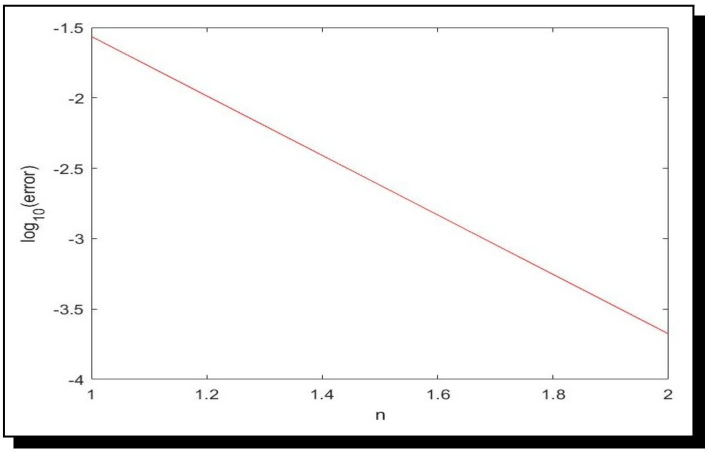

Figure 4. Algorithm 3.1

5. Concluding Remarks

In this manuscript, we presented a modified subgradiient extragradient method with inertial effects to solve the *Variational Inequalities* (VI) by incorporating the inertial terms in the modified subgradient extragradient method. The convergence result for Algorithm 3.1 presented under some standard assumptions. Also, the numerical results confirm the effectiveness of our proposed method.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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