Communications in Mathematics and Applications
Vol. 9, No. 4, pp. 725–735, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v9i4.1047

Research Article

Fixed Point Theorems on Soft *S*-Metric Spaces

Cigdem Gunduz Aras[1,*], Sadi Bayramov[2] and Vefa Cafarli[2]

[1] *Department of Mathematics, Kocaeli University, Kocaeli, 41380, Turkey*
[2] *Department of Algebra and Geometry, Baku State University, Baku, AZ 1148, Azerbaican*
***Corresponding author:** caras@kocaeli.edu.tr

Abstract. The aim to this paper is to define soft S-metric spaces and to investigate some important properties. In addition to, we prove some fixed point theorems of soft contractive mappings on soft S-metric spaces.

Keywords. Soft set; Soft S-metric space; Soft contractive mapping; Fixed point theorem

MSC. 54H25; 47H10

Received: August 5, 2018 **Accepted:** September 9, 2018

This article was submitted to "*5th International Congress on Fundamental and Applied Sciences*" (ICFAS 2018) organized by International University of Balkan, Skopje, Macedonia during June 18-22, 2018.

1. Introduction and Preliminaries

The metric space is one of the most important space in mathematics. There are various type of generalization of metric spaces. Sedghi *et al*. [13] initiated the notion of S-metric space which is different from other generalizations of the usual metric spaces. Several authors have studied the fixed point theory for functions on different metric spaces. Sedgi *et al*. [13] proved a fixed point theorem for a self-mapping on a complete S-metric space. A number of authors have defined contractive type mapping on a complete metric space which are generalizations of the well known Banach contraction, and which have the property that each such mapping has a unique fixed point in ([8], [15]).

Metric spaces wide area provide a powerful tool to the study of optimization and approximation theory, variational inequalities etc. Molodtsov [11] initiated a novel concept of soft set theory as a new mathematical tool for dealing with uncertainties, applications of soft set theory in other disciplines and real life problems was progressing rapidly, later on, the study of soft metric space which is based on soft point of soft sets was initiated by Das and Samanta [4]. Yazar *et al*. [16] examined some important properties of soft metric spaces and soft continuous mappings. They also proved some fixed point theorems of soft contractive mappings on soft metric spaces.

Topological structures of soft set have been studied by some authors. In [14], Shabir and Naz have initiated the study of soft topological spaces which are defined over an initial universe with a fixed set of parameters and showed that a soft topological space gives a parameterized family of topological spaces. Theoretical studies of soft topological spaces have also been researched by some authors in [2], [3], [5], [7], [10], [12], etc.

The purpose of this paper firstly is to contribute for investigating on soft $S-$metric space which is based on soft point of soft sets and give some of their properties. Secondly, we introduce contractive mappings on soft S-metric spaces and prove a common fixed point theorem for a self-mapping on complete soft S-metric spaces.

Throughout this paper, X denotes initial universe, E denotes the set of all parameters, $P(X)$ denotes the power set of X.

Definition 1 ([11]). A pair (F,E) is called a soft set over X, where F is a mapping given by $F: E \to P(X)$.
In other words, the soft set is a parameterized family of subsets of the set X. For $a \in E$, $F(a)$ may be considered as the set of a-elements of the soft set (F,E), or as the set of a-approximate elements of the soft set.

Definition 2 ([9]). A soft set (F,E) over X is said to be a null soft set denoted by Φ if for all $a \in E$, $F(a) = \varnothing$.

Definition 3 ([9]). A soft set (F,E) over X is said to be an absolute soft set denoted by $\widetilde{X}$ if for all $a \in E$, $F(a) = X$.

Definition 4 ([14]). Let $\widetilde{\tau}$ be the collection of soft sets over X, then $\widetilde{\tau}$ is said to be a soft topology on X if

(1) $\Phi, \widetilde{X}$ belongs to $\widetilde{\tau}$;

(2) the union of any number of soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$;

(3) the intersection of any two soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triplet $(X,\widetilde{\tau},E)$ is called a soft topological space over X.

Definition 5 ([14]). Let $(X,\widetilde{\tau},E)$ be a soft topological space over X, then members of $\widetilde{\tau}$ are said to be a soft open sets in X.

Definition 6 ([14])**.** Let $(X,\tilde{\tau},E)$ be a soft topological space over X. A soft set (F,E) over X is said to be a soft closed set in X, if its complement $(F,E)^c$ belongs to $\tilde{\tau}$.

Proposition 1 ([14])**.** *Let $(X,\tilde{\tau},E)$ be a soft topological space over X. Then the collection $\tilde{\tau}_a = \{F(a) : (F,E) \in \tilde{\tau}\}$ for each $a \in E$, defines a topology on X.*

Definition 7 ([14])**.** Let $(X,\tilde{\tau},E)$ be a soft topological space over X and (F,E) be a soft set over X. Then the soft closure of (F,E), denoted by $\overline{(F,E)}$, is the intersection of all soft closed super sets of (F,E). Clearly $\overline{(F,E)}$ is the smallest soft closed set over X which contains (F,E).

Definition 8 ([1], [4])**.** Let (F,E) be a soft set over X. The soft set (F,E) is called a soft point, denoted by (x_a,E), if for the element $a \in E$, $F(a) = \{x\}$ and $F(a') = \varnothing$ for all $a' \in E - \{a\}$ (briefly denoted by x_a).

It is obvious that each soft set can be expressed as a union of soft points. For this reason, to give the family of all soft sets on X it is sufficient to give only soft points on X.

Definition 9 ([1])**.** The soft point x_a is said to be belonging to the soft set (F,E), denoted by $x_a \tilde{\in} (F,E)$, if $x_a(a) \in F(a)$, i.e., $\{x\} \subseteq F(a)$.

Definition 10 ([4])**.** Let $\mathbb{R}$ be the set of all real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. It is denoted by (F,E). If (F,E) is a singleton soft set, then it will be called a soft real number and denoted by $\tilde{r},\tilde{s},\tilde{t}$ etc. Here $\tilde{r},\tilde{s},\tilde{t}$ will denote a particular type of soft real numbers such that $\tilde{r}(a) = r$, for all $a \in E$. $\tilde{0}$ and $\tilde{1}$ are the soft real numbers where $\tilde{0}(a) = 0$, $\tilde{1}(a) = 1$ for all $a \in E$, respectively.

The following definition is about a partial ordering on the set of soft real numbers.

Definition 11 ([4])**.** Let $\tilde{r},\tilde{s}$ be two soft real numbers, then the following statements hold:

(i) $\tilde{r} \tilde{\leq} \tilde{s}$, if $\tilde{r}(a) \leq \tilde{s}(a)$, for all $a \in E$,

(ii) $\tilde{r} \tilde{\geq} \tilde{s}$, if $\tilde{r}(a) \geq \tilde{s}(a)$, for all $a \in E$,

(iii) $\tilde{r} \tilde{<} \tilde{s}$, if $\tilde{r}(a) < \tilde{s}(a)$, for all $a \in E$,

(iv) $\tilde{r} \tilde{>} \tilde{s}$, if $\tilde{r}(a) > \tilde{s}(a)$, for all $a \in E$.

Definition 12 ([13])**.** Let X be a nonempty set and $S : X^3 \to [0,\infty)$ be a function satisfying the following conditions for all $x,y,z,t \in X$.

(1) $S(x,y,z) = 0$ if and only if $x = y = z$,

(2) $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$.

Then S is called an S-metric on X and the pair (X,S) is called an S-metric space.

In the next section, we give soft S-metric space and prove some fixed point theorems of soft contractive mappings on soft S-metric spaces. Let $\tilde{X}$ be the absolute soft set, E be a non-empty set of parameters and $SP(\tilde{X})$ be the collection of all soft points of $\tilde{X}$. Let $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers.

2. Contractive Mapping on Soft *S*-Metric Spaces

In this section, we introduce soft S-metric spaces and study its some important results. Later, we give some important concepts such as complete soft S-metric space, Cauchy sequence, soft continuous mapping on soft S-metric spaces. Let $\widetilde{X}$ be the absolute soft set, E be a non-empty set of parameters and $SP(\widetilde{X})$ be the collection of all soft points of $\widetilde{X}$. Let $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers.

Definition 13 ([6])**.** A soft S-metric on X is a mapping $S : SP(\widetilde{X})\times SP(\widetilde{X})\times SP(\widetilde{X}) \to \mathbb{R}(E)^*$ that satisfies the following conditions, for each soft points $x_a, y_b, z_c, u_d \in SP(\widetilde{X})$,

(S1) $S(x_a, y_b, z_c) \geq \widetilde{0}$,

(S2) $S(x_a, y_b, z_c) = \widetilde{0}$ if and only if $x_a = y_b = z_c$,

(S3) $S(x_a, y_b, z_c) \leq S(x_a, x_a, u_d) + S(y_b, y_b, u_d) + S(z_c, z_c, u_d)$.

Then the soft set $\widetilde{X}$ with a soft S-metric S is called a soft S-metric space and denoted by $(\widetilde{X}, S, E)$.

Definition 14. Let $(\widetilde{X}, S, E)$ be a soft S-metric space. A map $(f,\varphi) : (\widetilde{X}, S, E) \to (\widetilde{X}, S, E)$ is said to be a soft contraction mapping if there exists a soft real number $\widetilde{q} \in \mathbb{R}(E)$, $\widetilde{0} \leq \widetilde{q} < \widetilde{1}$ ($\mathbb{R}(E)$ denotes the soft real numbers set) such that

$$S((f,\varphi)(x_a),(f,\varphi)(x_a),(f,\varphi)(y_b)) \leq \widetilde{q}S(x_a, x_a, y_b)$$

for all $x_a, y_b \in SP(\widetilde{X})$.

Note that a soft contraction mapping is a soft continuous mapping because if $x^n_{a_n} \to x_a$ in the above condition we get $(f,\varphi)(x^n_{a_n}) \to (f,\varphi)(x_a)$.

Remark 1. Let $(\widetilde{X}, S, E)$ be a soft S-metric space. It is clear that every soft S-metric space is a family of parameterized S-metric spaces. For $a \in E$, (X, S_a) is an S-metric space.

Proposition 2. *Let $(\widetilde{X}, S, E)$ be a soft S-metric space. If $(f,\varphi) : (\widetilde{X}, S, E) \to (\widetilde{X}, S, E)$ is a soft contraction mapping, then $f_a : (X, S_a) \to (X, S_{\varphi(a)})$ is a contraction mapping in S-metric space, for each $a \in E$.*

The following example shows that converse of Proposition 2 does not hold.

Example 1. Let $E = \mathbb{R}$ be a parameter set and $X = \mathbb{R}^2$. Consider usual metrics on this sets and define soft S-metric on $SP(\widetilde{X})$ by $S(x_a, y_b, z_c) = |b + c - 2a| + |b - c| + \|y + z - 2x\| + \|y - z\|$. Then if we define the soft mapping $(f,\varphi) : (\widetilde{X}, S, E) \to (\widetilde{X}, S, E)$ as follows $(f,\varphi)(x_a) = \left(\frac{1}{7}x\right)_{5a}$, then

$$\begin{aligned} S((f,\varphi)(0,1)_2,(f,\varphi)(0,1)_2,(f,\varphi)(1,0)_1) &= S\left(\left(0,\frac{1}{7}\right)_{10},\left(0,\frac{1}{7}\right)_{10},\left(\frac{1}{7},0\right)_5\right) \\ &= |-5| + |5| + \left\|\left(\frac{1}{7},-\frac{1}{7}\right)\right\| + \left\|\left(-\frac{1}{7},\frac{1}{7}\right)\right\| \\ &= 10 + 2\frac{\sqrt{2}}{7}, \end{aligned}$$

$$S((0,1)_2,(0,1)_2,(1,0)_1) = 2 + 2\sqrt{2}.$$

Then since $10+2\frac{\sqrt{2}}{7} > 2+2\sqrt{2}$, we see that the soft mapping (f,φ) is not a soft contraction mapping. But the mapping $f_a:(X,S_a)\to(X,S_{5a})$ by $f_a(x)=\frac{1}{7}x$ is a contraction mapping for all $a\in E$.

For notational purpose we define $(f,\varphi)^n(x_a)$, $x_a\in SP(\widetilde{X})$ and $n\in\{0,1,2,\dots\}$, inductively by $(f,\varphi)^0(x_a)=x_a$ and $(f,\varphi)^{n+1}(x_a)=(f,\varphi)((f,\varphi)^n(x_a))$.

Theorem 1. *Let $(\widetilde{X},S,E)$ be a complete soft S-metric space and $(f,\varphi):(\widetilde{X},S,E)\to(\widetilde{X},S,E)$ be a soft contraction mapping. Then (f,φ) has a unique fixed soft point $u_c\in \mathrm{SP}(\widetilde{X})$. Moreover, for any $x_a\in\mathrm{SP}(\widetilde{X})$, we have $\lim_{n\to\infty}(f,\varphi)^n(x_a)=u_c$ with*

$$S((f,\varphi)^n(x_a),(f,\varphi)^n(x_a),u_c)\le\frac{2\widetilde{q}}{\widetilde{1}-\widetilde{q}}S(x_a,x_a,(f,\varphi)(x_a)).$$

Proof. Let $x_a\in SP(\widetilde{X})$ be any soft point. First we show that $\{(f,\varphi)^n(x_a)\}_{n=\overline{0,\infty}}$ is a Cauchy sequence. Since (f,φ) is a soft contraction mapping, we get by induction

$$\begin{aligned}S((f,\varphi)^n(x_a),(f,\varphi)^n(x_a),(f,\varphi)^{n+1}(x_a))&\le\widetilde{q}S((f,\varphi)^{n-1}(x_a),(f,\varphi)^{n-1}(x_a),(f,\varphi)^n(x_a))\\&\le\widetilde{q}^2S((f,\varphi)^{n-2}(x_a),(f,\varphi)^{n-2}(x_a),(f,\varphi)^{n-1}(x_a))\\&\le\cdots\le\widetilde{q}^nS(x_a,x_a,(f,\varphi)(x_a)).\end{aligned}$$

Thus for $m\ge n$, we have

$$\begin{aligned}S((f,\varphi)^n(x_a),(f,\varphi)^n(x_a),(f,\varphi)^m(x_a))&\le 2S((f,\varphi)^n(x_a),(f,\varphi)^n(x_a),(f,\varphi)^{n+1}(x_a))\\&\quad+S((f,\varphi)^{n+1}(x_a),(f,\varphi)^{n+1}(x_a),(f,\varphi)^m(x_a))\\&\le\cdots\le 2\sum_{i=n}^{m-2}S((f,\varphi)^i(x_a),(f,\varphi)^i(x_a),(f,\varphi)^{i+1}(x_a))\\&\quad+S((f,\varphi)^{m-1}(x_a),(f,\varphi)^{m-1}(x_a),(f,\varphi)^m(x_a))\\&\le 2\sum_{i=n}^{m-2}\widetilde{q}^iS(x_a,x_a,(f,\varphi)(x_a))+\widetilde{q}^{m-1}S(x_a,x_a,(f,\varphi)(x_a))\\&\le 2\widetilde{q}^nS(x_a,x_a,(f,\varphi)(x_a))[1+\widetilde{q}+\widetilde{q}^2+\cdots]\\&\le\frac{2\widetilde{q}}{\widetilde{1}-\widetilde{q}}S(x_a,x_a,(f,\varphi)(x_a)).\end{aligned}$$

That is for $m>n$,

$$S((f,\varphi)^n(x_a),(f,\varphi)^n(x_a),(f,\varphi)^m(x_a))\le\frac{2\widetilde{q}}{\widetilde{1}-\widetilde{q}}S(x_a,x_a,(f,\varphi)(x_a)) \tag{1}$$

Since $\widetilde{q}<\widetilde{1},\{(f,\varphi)^n(x_a)\}_{n=\overline{0,\infty}}$ is a Cauchy sequence, by the completeness of $(\widetilde{X},S,E)$, there exists a soft point $u_c\in SP(\widetilde{X})$ such that $\lim_{n\to\infty}(f,\varphi)^n(x_a)=u_c$. Moreover, the continuity of (f,φ) yields

$$u_c=\lim_{n\to\infty}(f,\varphi)^{n+1}(x_a)=\lim_{n\to\infty}(f,\varphi)((f,\varphi)^n(x_a))=(f,\varphi)(u_c).$$

Now, we show the uniqueness. Suppose that there exist $u_c,v_b\in SP(\widetilde{X})$ with $(f,\varphi)(u_c)=u_c$ and $(f,\varphi)(v_b)=v_b$. Then

$$\begin{aligned}S(u_c,u_c,v_b)&=S((f,\varphi)(u_c),(f,\varphi)(u_c)(f,\varphi)(v_b))\\&\le\widetilde{q}S(u_c,u_c,v_b)\end{aligned}$$

and therefore $S(u_c,u_c,v_b)=\widetilde{0}$. Hence u_c is a fixed soft point of (f,φ).

Finally, letting $m\to\infty$ in (1) we obtain

$$S((f,\varphi)^n(x_a),(f,\varphi)^n(x_a),u_c)\le\frac{2\widetilde{q}}{\widetilde{1}-\widetilde{q}}S(x_a,x_a,(f,\varphi)(x_a)).$$ □

Example 2. Let $E=\mathbb{N}$ be a parameter set and $X=\mathbb{R}$. Then

$$S(x_a,y_b,z_c)=|a-c|+|b-c|+|x-z|+|y-z|$$

is a soft S-metric. Define a soft mapping

$$(f,\varphi):(\widetilde{X},S,E)\to(\widetilde{X},S,E)$$

as follows $(f,\varphi)(x_a)=\left(\frac{1}{2}\sin x\right)_1$, where $f(x)=\frac{1}{2}\sin x$ and $\varphi(a)=1$ are constant mappings. We have

$$\begin{aligned}S((f,\varphi)(x_a),(f,\varphi)(x_a),(f,\varphi)(y_b))&=S\left(\left(\frac{1}{2}\sin x\right)_1,\left(\frac{1}{2}\sin x\right)_1,\left(\frac{1}{2}\sin y\right)_1\right)\\&=\left|\frac{1}{2}(\sin x-\sin y)\right|+\left|\frac{1}{2}(\sin x-\sin y)\right|\\&\le\frac{1}{2}(|x-y|+|x-y|)\\&\le\frac{1}{2}(|a-b|+|a-b|+|x-y|+|x-y|)\\&=\frac{1}{2}S(x_a,x_a,y_b)\end{aligned}$$

for every $x_a,y_b\in SP(\widetilde{X})$. Furthermore, for any $x_a\in SP(\widetilde{X})$, we have $\lim_{n\to\infty}(f,\varphi)^n(x_a)=0_1$ with

$$S((f,\varphi)^n(x_a),(f,\varphi)^n(x_a),0_1)\le\frac{2\widetilde{q}}{\widetilde{1}-\widetilde{q}}S(x_a,x_a,(f,\varphi)(x_a)),\quad \widetilde{q}=\frac{\widetilde{1}}{2}.$$

It follows that all condition of Theorem 1 hold and there exists $u_c=0_1\in SP(\widetilde{X})$ such that $(f,\varphi)(0_1)=0_1$.

Definition 15. Let $(\widetilde{X},S,E)$ be a soft S-metric space. $(\widetilde{X},S,E)$ is called a soft sequential compact space, if every soft sequence has a soft subsequence that converges in $(\widetilde{X},S,E)$.

Theorem 2. *Let $(\widetilde{X},S,E)$ be a soft sequential compact soft S-metric space with $(f,\varphi):(\widetilde{X},S,E)\to(\widetilde{X},S,E)$ satisfying*

$$S((f,\varphi)(x_a),(f,\varphi)(x_a),(f,\varphi)(y_b))<S(x_a,x_a,y_b)$$

for every $x_a,y_b\in \mathrm{SP}(\widetilde{X})$ and $x_a\ne y_b$. Then (f,φ) has a unique soft fixed point in $(\widetilde{X},S,E)$.

Proof. The uniqueness is clear. To show the existence, notice that the mapping $x_a\mapsto S(x_a,x_a,(f,\varphi)(x_a))$ attains its minimum, say that $x_b\in SP(\widetilde{X})$. We have $x_b=(f,\varphi)(x_b)$ since otherwise

$$\begin{aligned}S((f,\varphi)((f,\varphi)(x_b)),(f,\varphi)((f,\varphi)(x_b)),(f,\varphi)(x_b))&<S((f,\varphi)(x_b),(f,\varphi)(x_b),x_b)\\&=S(x_b,x_b,(f,\varphi)(x_b))\end{aligned}$$

which is a contradiction. □

Theorem 3. *Let $(\widetilde{X},S,E)$ be a complete soft S-metric space and let $B_S(x_a^0,\widetilde{r}) = \{x_b \in \mathrm{SP}(\widetilde{X}) : S(x_a^0,x_a^0,x_b) < \widetilde{r}\}$, where $x_a^0 \in \mathrm{SP}(\widetilde{X})$ and $\widetilde{r} > \widetilde{0}$. Suppose that $(f,\varphi): B_S(x_a^0,\widetilde{r}) \to (\widetilde{X},S,E)$ is a soft contraction mapping with*

$$S((f,\varphi)(x_a^0),(f,\varphi)(x_a^0),x_a^0) < (\widetilde{1}-\widetilde{q})\frac{\widetilde{r}}{2}.$$

Then (f,φ) has a unique fixed soft point in $B_S(x_a^0,\widetilde{r})$.

Proof. There exists $\widetilde{r}_0$ with $\widetilde{0} \le \widetilde{r}_0 < \widetilde{r}$ such that $S((f,\varphi)(x_a^0),(f,\varphi)(x_a^0),x_a^0) < (\widetilde{1}-\widetilde{q})\frac{\widetilde{r}_0}{2}$. We will show that

$$(f,\varphi): \overline{B_S(x_a^0,\widetilde{r})} \to \overline{B_S(x_a^0,\widetilde{r})}.$$

If $x_b \in \overline{B_S(x_a^0,\widetilde{r})}$, then

$$\begin{aligned} S(x_a^0,x_a^0,(f,\varphi)(x_b)) &\le 2S(x_a^0,x_a^0,(f,\varphi)(x_a^0)) + S((f,\varphi)(x_a^0),(f,\varphi)(x_a^0),(f,\varphi)(x_b)) \\ &\le 2(\widetilde{1}-\widetilde{q})\frac{\widetilde{r}_0}{2} + \widetilde{q}S(x_a^0,x_a^0,x_b) \le \widetilde{r}_0. \end{aligned}$$

Hence, we can apply Theorem 1, the soft mapping (f,φ) has a unique soft fixed point in $\overline{B_S(x_a^0,\widetilde{r}_0)} \subset B_S(x_a^0,\widetilde{r})$. □

Now, let $E = \mathbb{R}$ be a parameter set and $(X,\|\cdot\|)$ be a Banach space. Then, we define the following operations in $SP(\widetilde{X})$

(1) $x_a + y_b = (x+y)_{a+b}$,

(2) $k \cdot x_a = (k \cdot x)_{k\cdot a}$.

Then $SP(\widetilde{X})$ is a soft vector space. We define norm operation in $SP(\widetilde{X})$ by $\|x_a\| = |a| + \|x\|$. It is clear that

$$S(x_a,y_b,z_c) = \|x_a - y_b\| + \|y_b - z_c\|$$

is a soft S-metric in $SP(\widetilde{X})$. More general results will be presented in the next theorem.

Theorem 4. *Let $(\widetilde{X},S,\mathbb{R})$ be a complete soft S-metric space and $\overline{B_S(\theta_0,\widetilde{r})}$ be the soft closed ball of radius $\widetilde{r} > \widetilde{0}$. If*

$$(F,1_{\mathbb{R}}): \overline{B_S(\theta_0,\widetilde{r})} \to (\widetilde{X},S,\mathbb{R})$$

is a soft contraction mapping and $(F,1_{\mathbb{R}})(\partial\overline{B_S(\theta_0,\widetilde{r})}) \subset \overline{B_S(\theta_0,\widetilde{r})}$, then $(F,1_{\mathbb{R}})$ has a unique soft point in $\overline{B_S(\theta_0,\widetilde{r})}$.

Proof. For soft point $x_a \in \overline{B_S(\theta_0,\widetilde{r})}$, consider $(G,1_{\mathbb{R}})(x_a) = \frac{x_a + (F,1_{\mathbb{R}})(x_a)}{2}$. We first show that

$$(G,1_{\mathbb{R}}): \overline{B_S(\theta_0,\widetilde{r})} \to \overline{B_S(\theta_0,\widetilde{r})}.$$

To see this, let $x_a^* = \left(\frac{\widetilde{r}}{\|x\|}x\right)_a$ where $x_a \in \overline{B_S(\theta_0,\widetilde{r})}$. Then we have

$$\begin{aligned} S((F,1_{\mathbb{R}})(x_a),(F,1_{\mathbb{R}})(x_a),(F,1_{\mathbb{R}})(x_a^*)) &= \|F(x) - F(x^*)\| \le \widetilde{q}S(x_a,x_a,x_a^*) \\ &= \widetilde{q}\|x_a - x_a^*\| = \widetilde{q}\|x - x^*\| \\ &= \widetilde{q}\left\|x - \frac{\widetilde{r}}{\|x\|}x\right\| = \widetilde{q}(\widetilde{r} - \|x\|). \end{aligned}$$

Hence

$$\begin{aligned}\|F(x)\| &\le \|F(x^*)\| + \|F(x)-F(x^*)\| \\ &\le \widetilde{r} + \widetilde{q}(\widetilde{r} - \|x\|) \\ &< 2\widetilde{r} - \|x\|.\end{aligned}$$

Then for $x_a \in \overline{B_S(\theta_0, \widetilde{r})}$,

$$\begin{aligned}\|(G,1_{\mathbb{R}})(x_a)\| &= \left\| \frac{x_a + (F,1_{\mathbb{R}})(x_a)}{2} \right\| \\ &\le \frac{\|x_a\| + \|(F,1_{\mathbb{R}})(x_a)\|}{2} \le \widetilde{r}.\end{aligned}$$

It is seen that $(G,1_{\mathbb{R}}) : \overline{B_S(\theta_0,\widetilde{r})} \to \overline{B_S(\theta_0,\widetilde{r})}$ is a soft contraction mapping because

$$\begin{aligned}\|(G,1_{\mathbb{R}})(x_a) - (G,1_{\mathbb{R}})(y_b)\| &= \left\| \frac{x_a + (F,1_{\mathbb{R}})(x_a)}{2} - \frac{y_b + (F,1_{\mathbb{R}})(y_b)}{2} \right\| \\ &\le \frac{\|x-y\| + \widetilde{q}\|x-y\|}{2} = \left(\frac{\widetilde{1}+\widetilde{q}}{2}\right)\|x-y\|.\end{aligned}$$

Theorem 1 implies that $(G,1_{\mathbb{R}})$ has a unique fixed soft point in $u_c \in \overline{B_S(\theta_0,\widetilde{r})}$ and so $u_c = (F,1_{\mathbb{R}})(u_c)$. □

Theorem 5. *Let $(\widetilde{X}, S, E)$ be a complete soft S-metric space. Suppose that the soft mapping $(f,\varphi) : (\widetilde{X},S,E) \to (\widetilde{X},S,E)$ satisfies the soft contractive condition*

$$S((f,\varphi)(x_a),(f,\varphi)(x_a),(f,\varphi)(y_b)) \le \widetilde{\alpha}[S((f,\varphi)(x_a),(f,\varphi)(x_a),x_a) + S(((f,\varphi)(y_b),(f,\varphi)(y_b),y_b))]$$

for every $x_a, y_b \in \mathrm{SP}(\widetilde{X})$, where $\widetilde{\alpha} \in [\widetilde{0}, \widetilde{\tfrac{1}{2}})$ is a soft constant real number. Then (f,φ) has a unique fixed soft point in $\mathrm{SP}(\widetilde{X})$.

Proof. Choose x_a be any soft point in $SP(\widetilde{X})$. From Theorem 1, we set the soft sequence $\{x^n_{a_n}\}$ as follows

$$x^1_{a_1} = (f,\varphi)(x_a) = (f(x))_{\varphi(a)}, \dots, x^{n+1}_{a_{n+1}} = (f,\varphi)(x^n_{a_n}) = (f(x^n))_{\varphi(a_n)}, \dots$$

We have

$$\begin{aligned}S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^n_{a_n}) &= S((f,\varphi)(x^n_{a_n}),(f,\varphi)(x^n_{a_n}),(f,\varphi)(x^{n-1}_{a_{n-1}})) \\ &\le \widetilde{\alpha}[S((f,\varphi)(x^n_{a_n}),(f,\varphi)(x^n_{a_n}),x^n_{a_n}) + S((f,\varphi)(x^{n-1}_{a_{n-1}}),(f,\varphi)(x^{n-1}_{a_{n-1}}),x^{n-1}_{a_{n-1}})] \\ &= \widetilde{\alpha}[S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^n_{a_n}) + S(x^n_{a_n}, x^n_{a_n}, x^{n-1}_{a_{n-1}})] \\ &\le \frac{\widetilde{\alpha}}{\widetilde{1}-\widetilde{\alpha}} S(x^n_{a_n}, x^n_{a_n}, x^{n-1}_{a_{n-1}}) \\ &= \widetilde{h} S(x^n_{a_n}, x^n_{a_n}, x^{n-1}_{a_{n-1}}),\end{aligned}$$

where $\widetilde{h} = \frac{\widetilde{\alpha}}{\widetilde{1}-\widetilde{\alpha}}$. For $n > m$,

$$\begin{aligned}S(x^n_{a_n}, x^n_{a_n}, x^m_{a_m}) &\le 2S(x^n_{a_n}, x^n_{a_n}, x^{n+1}_{a_{n+1}}) + S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^m_{a_m}) \\ &\le \dots \le 2\sum_{i=n}^{m-2} S(x^i_{a_i}, x^i_{a_i}, x^{i+1}_{a_{i+1}}) + S(x^{m-1}_{a_{m-1}}, x^{m-1}_{a_{m-1}}, x^m_{a_m}) \\ &\le 2\sum_{i=n}^{m-2} \widetilde{h}^i S(x_a, x_a, x^1_{a_1}) + \widetilde{h}^{m-1} S(x_a, x_a, x^1_{a_1})\end{aligned}$$

$$\leq \frac{2\widetilde{h}^n}{\widetilde{1}-\widetilde{h}} S(x_a, x_a, x^1_{a_1}).$$

We get $S(x^n_{a_n}, x^n_{a_n}, x^m_{a_m}) \leq \frac{2\widetilde{h}^n}{\widetilde{1}-\widetilde{h}} S(x_a, x_a, x^1_{a_1})$. This implies $S(x^n_{a_n}, x^n_{a_n}, x^m_{a_m}) \to \widetilde{0}$. Hence $\{x^n_{a_n}\}$ is a Cauchy sequence, by the completeness of $(\widetilde{X}, S, E)$, $\{x^n_{a_n}\}$ converges. Suppose that $x^n_{a_n} \to x^*_c$. Since

$$\begin{aligned} S(x^*_c, x^*_c, (f,\varphi)(x^*_c)) &\leq 2S((f,\varphi)(x^*_c), (f,\varphi)(x^*_c), (f,\varphi)(x^n_{a_n})) + S((f,\varphi)(x^n_{a_n}), (f,\varphi)(x^n_{a_n}), x^*_c) \\ &\leq \widetilde{\alpha}[2S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^n_{a_n}) + 2S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^*_c)] + S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^*_c) \\ &\leq \frac{2\widetilde{\alpha}}{\widetilde{1}-\widetilde{\alpha}} [S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^n_{a_n}) + S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^*_c)] \to \widetilde{0}. \end{aligned}$$

This implies $(f,\varphi)(x^*_c) = x^*_c$. □

Theorem 6. *Let $(\widetilde{X}, S, E)$ be a complete soft S-metric space. Suppose that the soft mapping $(f,\varphi) : (\widetilde{X}, S, E) \to (\widetilde{X}, S, E)$ satisfies the soft contraction condition*

$$S((f,\varphi)(x_a), (f,\varphi)(x_a), (f,\varphi)(y_b)) \leq \widetilde{\alpha}[S((f,\varphi)(x_a), (f,\varphi)(x_a), y_b) + S(((f,\varphi)(y_b), (f,\varphi)(y_b), x_a))]$$

for every $x_a, y_b \in \mathrm{SP}(\widetilde{X})$, where $\widetilde{\alpha} \in [\widetilde{0}, \widetilde{\frac{1}{2}})$ is a soft constant real number. Then (f,φ) has a unique fixed soft point in $\mathrm{SP}(\widetilde{X})$.

Proof. Choose x_a be any soft point in $SP(\widetilde{X})$. From Theorem 1, we set the soft sequence $\{x^n_{a_n}\}$ as follows $x^1_{a_1} = (f,\varphi)(x_a) = (f(x))_{\varphi(a)}, \ldots, x^{n+1}_{a_{n+1}} = (f,\varphi)(x^n_{a_n}) = (f(x^n))_{\varphi(a_n)}, \ldots$.

We have

$$\begin{aligned} S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^n_{a_n}) &= S((f,\varphi)(x^n_{a_n}), (f,\varphi)(x^n_{a_n}), (f,\varphi)(x^{n-1}_{a_{n-1}})) \\ &\leq \widetilde{\alpha}[S((f,\varphi)(x^n_{a_n}), (f,\varphi)(x^n_{a_n}), x^n_{a_n}) + S((f,\varphi)(x^{n-1}_{a_{n-1}}), (f,\varphi)(x^{n-1}_{a_{n-1}}), x^{n-1}_{a_{n-1}})] \\ &\leq \widetilde{\alpha}[S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^n_{a_n}) + S(x^n_{a_n}, x^n_{a_n}, x^{n-1}_{a_{n-1}})]. \end{aligned}$$

So,

$$S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^n_{a_n}) \leq \frac{\widetilde{\alpha}}{\widetilde{1}-\widetilde{\alpha}} S(x^n_{a_n}, x^n_{a_n}, x^{n-1}_{a_{n-1}}) = \widetilde{h} S(x^n_{a_n}, x^n_{a_n}, x^{n-1}_{a_{n-1}}),$$

where $\widetilde{h} = \frac{\widetilde{\alpha}}{\widetilde{1}-\widetilde{\alpha}}$. For $n > m$,

$$\begin{aligned} S(x^n_{a_n}, x^n_{a_n}, x^m_{a_m}) &\leq 2S(x^n_{a_n}, x^n_{a_n}, x^{n+1}_{a_{n+1}}) + S(x^{n+1}_{a_{n+1}}, x^{n+1}_{a_{n+1}}, x^m_{a_m}) \\ &\leq \cdots \leq 2\sum_{i=n}^{m-2} S(x^i_{a_i}, x^i_{a_i}, x^{i+1}_{a_{i+1}}) + S(x^{m-1}_{a_{m-1}}, x^{m-1}_{a_{m-1}}, x^m_{a_m}) \\ &\leq 2\sum_{i=n}^{m-2} \widetilde{h}^i S(x_a, x_a, x^1_{a_1}) + \widetilde{h}^{m-1} S(x_a, x_a, x^1_{a_1}) \\ &\leq 2\sum_{i=n}^{m-2} \widetilde{h}^i S(x_a, x_a, x^1_{a_1}) + \widetilde{h}^{m-1} S(x_a, x_a, x^1_{a_1}) \\ &\leq \frac{2\widetilde{h}^n}{\widetilde{1}-\widetilde{h}} S(x_a, x_a, x^1_{a_1}). \end{aligned}$$

We get $S(x^n_{a_n}, x^n_{a_n}, x^m_{a_m}) \leq \frac{2\widetilde{h}^n}{\widetilde{1}-\widetilde{h}} S(x_a, x_a, x^1_{a_1})$. This implies $S(x^n_{a_n}, x^n_{a_n}, x^m_{a_m}) \to \widetilde{0}$. Hence $\{x^n_{a_n}\}$ is a Cauchy sequence, by the completeness of $(\widetilde{X}, S, E)$, $\{x^n_{a_n}\}$ converges. Suppose that $x^n_{a_n} \to x^*_c$.

Since

$$\begin{aligned}S(x^*_c,x^*_c,(f,\varphi)(x^*_c)) &\le 2S((f,\varphi)(x^*_c),(f,\varphi)(x^*_c),(f,\varphi)(x^n_{a_n}))+S((f,\varphi)(x^n_{a_n}),(f,\varphi)(x^n_{a_n}),x^*_c)\\ &\le \widetilde{\alpha}[2S(x^{n+1}_{a_{n+1}},x^{n+1}_{a_{n+1}},x^n_{a_n})+2S(x^{n+1}_{a_{n+1}},x^{n+1}_{a_{n+1}},x^*_c)]+S(x^{n+1}_{a_{n+1}},x^{n+1}_{a_{n+1}},x^*_c)\\ &\le \frac{2\widetilde{\alpha}}{\widetilde{1}-\widetilde{\alpha}}[S(x^{n+1}_{a_{n+1}},x^{n+1}_{a_{n+1}},x^n_{a_n})+S(x^{n+1}_{a_{n+1}},x^{n+1}_{a_{n+1}},x^*_c)]\to\widetilde{0}.\end{aligned}$$

This implies $(f,\varphi)(x^*_c)=x^*_c$. □

Remark 2. The following example shows that if $(f,\varphi):(\widetilde{X},S,E)\to(\widetilde{X},S,E)$ is a soft contraction mapping, then f or φ may not be a contraction mappings.

Example 3. Let $E=[1,\infty)$ be a parameter set and $X=\mathbb{R}$. Consider metrics $d_1(x,y)=\min\{1,|x-y|\}$, $d(x,y)=|x-y|$ on this sets, respectively. Define soft S-metric on $SP(\widetilde{X})$ by

$$S(x_a,y_b,z_c)=\frac{1}{2}(d_1(a,b)+d_1(b,c))+d(x,y)+d(y,z).$$

Let the mappings $\varphi:[1,\infty)\to[1,\infty)$ and $f:\mathbb{R}\to\mathbb{R}$ are defined as $\varphi(a)=a+\frac{1}{a}$ and $f(x)=\frac{1}{10}x$, respectively. Here, it is obvious that $\varphi:[1,\infty)\to[1,\infty)$ is not a contraction mapping with the defined metric $d_1(x,y)=\min\{1,|x-y|\}$. But (f,φ) is a soft contraction mapping on $SP(\widetilde{X})$. Indeed,

$$\begin{aligned}S((f,\varphi)(x_a),(f,\varphi)(x_a),(f,\varphi)(y_b)) &= S\left(\left(\frac{1}{10}x\right)_{a+\frac{1}{a}},\left(\frac{1}{10}x\right)_{a+\frac{1}{a}},\left(\frac{1}{10}y\right)_{b+\frac{1}{b}}\right)\\ &=\frac{1}{10}|x-y|+\frac{1}{2}d_1\left(a+\frac{1}{a},b+\frac{1}{b}\right)\\ &=\frac{1}{10}|x-y|+\frac{1}{2}\min\left\{\left|a+\frac{1}{a}-b-\frac{1}{b}\right|,1\right\}\\ &=\frac{1}{10}|x-y|+\frac{1}{2}\min\left\{|a-b|\left|1-\frac{1}{ab}\right|,1\right\}\\ &\le\frac{1}{10}|x-y|+\frac{1}{2}\min\{|a-b|,1\}\\ &=\frac{1}{10}|x-y|+\frac{1}{2}d_1(a,b)\\ &\le\frac{3}{4}(|x-y|+d_1(a,b))\\ &=\frac{3}{4}S(x_a,x_a,y_b).\end{aligned}$$

3. Conclusion

We have introduced soft S-metric space which is based on soft point of soft sets and give some of their properties. In addition to, we define the concept of soft contractive mappings on soft S-metric space and examine some important properties. Finally, we prove some fixed point theorems of soft contractive mappings on soft S-metric spaces.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. Bayramov and C. Gunduz, Soft locally compact spaces and soft paracompact spaces, *J. Math. System Sci.* **3** (2013), 122 – 130, DOI: 10.17265/2159-5291/2013.03.002.

[2] S. Bayramov and C. Gunduz, A new approach to separability and compactness in soft topological spaces, *TWMS Journal of Pure and Applied Mathematics* **9**(21) (2018), 82 – 93, `http://iam.bsu.edu.az/`

[3] S. Bayramov, C. Gunduz and L. Mdzinarishvili, Singular homology theory in the category of soft topological spaces, *Georgian Math. J.* **22**(4) (2015), 457 – 467, DOI: 10.1515/gmj-2015-0042.

[4] S. Das and S.K. Samanta, Soft metric, *Ann. Fuzzy Math. Inform.* **6**(1) (2013), 77 – 94, `http://www.afmi.or.kr`.

[5] C. Gunduz Aras and S. Bayramov, On the Tietze extension theorem in soft topological spaces, *Proc. of Inst. Math. & Mech.* **43**(1) (2017), 105 – 115, `http://www.proc.imm.az/`.

[6] C. Gunduz Aras, S. Bayramov and V. Cafarli, A study on soft S-metric spaces, *Commun. Math. & Appl.* **9**(4) (2018), 713 – 723, DOI: 10.26713/cma.v9i4.975.

[7] S. Hussain and B. Ahmad, Some properties of soft topological spaces, *Comput. Math. Appl.* **62** (2011), 4058 – 4067, DOI: 10.1016/j.camwa.2011.09.051.

[8] H. Long-Guang and Z. Xian, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.* **332** (2007), 1468 – 1476, DOI: 10.1016/j.jmaa.2005.03.087.

[9] P.K. Maji, R. Biswas and A.R. Roy, Soft set theory, *Comput. Math. Appl.* **45** (2003), 555 – 562, PII: SO898-1221(03)00016-6.

[10] W.K. Min, A note on soft topological spaces, *Comput. Math. Appl.* **62** (2011), 3524 – 3528, DOI: 10.1016/j.camwa.2011.08.068.

[11] D. Molodtsov, Soft set theory-first results, *Comput. Math. Appl.* **37** (1999), 19 – 31, PII: S0898-1221(99)00056-5.

[12] T.Y. Ozturk and S. Bayramov, Soft mapping spaces, *The Scientific World Journal* **2014** (2014), Article ID 307292, 8 pages, DOI: 10.1155/2014/307292.

[13] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorem in S-metric spaces, *Mat. Vesnik* **64** (2012), 258 – 266, `https://www.emis.de/journals/MV/123/mv12309.pdf`.

[14] M. Shabir and M. Naz, On soft topological spaces, *Comput. Math. Appl.* **61** (2011), 1786 – 1799, DOI: 10.1016/j.camwa.2011.02.006.

[15] B.E. Rhoades, A comparison of various definition of contractive mappings, *Trans. Amer. Math. Soc.* **226** (1977), 257 – 290, DOI: 10.2307/1997954.

[16] M.I. Yazar, C. Gunduz Aras and S. Bayramov, Fixed point theorems of soft contractive mappings, *Filomat* **30**(2) (2016), 269 – 279, DOI: 10.2298/FIL1602269Y.